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ABSTRACT

As one of three audiovisual media in the U. S. Naval Academy Self-Paced Physics Course 27 topics relating to mechanics, electricity, and magnetism are presented in this volume for enriching and supplementary purposes. Each topic is primarily composed of figures and formulas associated with explanatory statements. Terminal behavioral objectives and directions for reaching subsequent study guides are incorporated at the end of the topic. The material is designed to optimize and individualize the student learning process. (Related documents are SE 016 065 through SE 016 088 and ED 062 123 through ED 062 125.) (CC)

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ILLUSTRATED TEXTS

(MASTER SET)

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SE 016 080

ILLUSTRATED BOOKS

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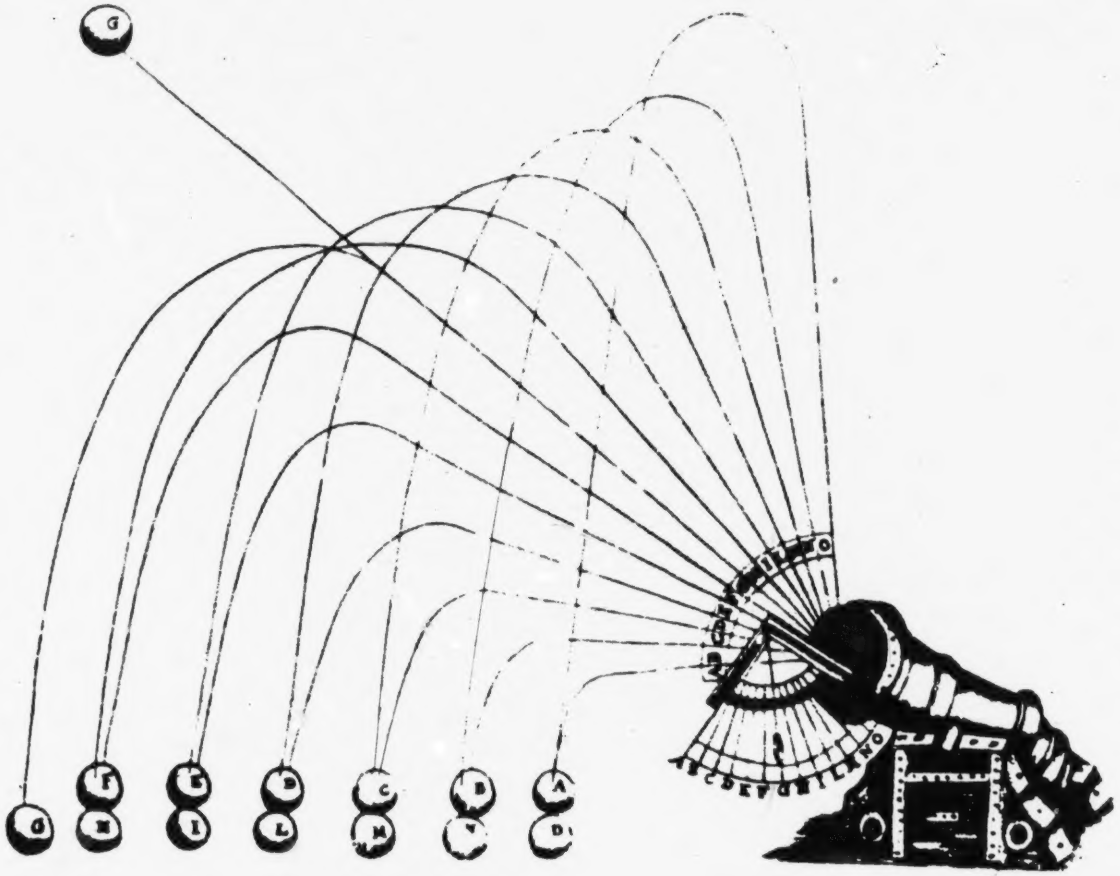
PROJECTILE
MOTION

Man's slow and tortuous climb out of the primeval ooze probably began with the invention of the club, but his progress was unquestionably accelerated when he learned to throw stones with some degree of accuracy. Today he is still throwing things. The only difference between modern man and his very early ancestors in this respect is that the things being thrown today are considerably more lethal than stones and that he no longer uses his arm muscles to throw them!

As early as the 16th century, much attention was already being given to accuracy of "throwing". For example, as shown in Figure 1, artillerymen were beginning to apply some mathematics to the aiming of their cannon. In this illustration, the cannoneer is being taught how to use a quadrant to help him achieve the desired trajectory by selecting the proper angle of elevation for the cannon.

Of course, we can do a lot better today. We understand projectile motion; we know how to apply mathematics to the motion so that we can predict how the projectile will move, how high it will rise in the air, and how far down-range it will go.


G
F
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ABCDEFGHILNO

FIGURE ①

An understanding of projectile motion must start with fundamentals (Figure 2). A gun is placed so that its muzzle makes an angle θ with the ground considered here to be perfectly horizontal. The projectile is fired so that it leaves the muzzle with an initial velocity $\underline{v}$. Its subsequent motion will depend to a great extent on the length of time it will remain in the air before returning to the ground, hence this is the first consideration to be worked out.

To calculate the total time of flight, it is first necessary to determine the time needed for the projectile to reach the highest point in its flight. This approach requires that the initial velocity $\underline{v}$ at an angle θ to the ground be resolved into its horizontal and vertical components, v_x and v_y respectively. (Figure 3).

The horizontal component v_x is trigonometrically related to the initial velocity $\underline{v}$ by the cosine of angle θ. Hence, $v_x = v \cos \theta$. Similarly, $v_y = v \sin \theta$. Refer now to Figure 4.

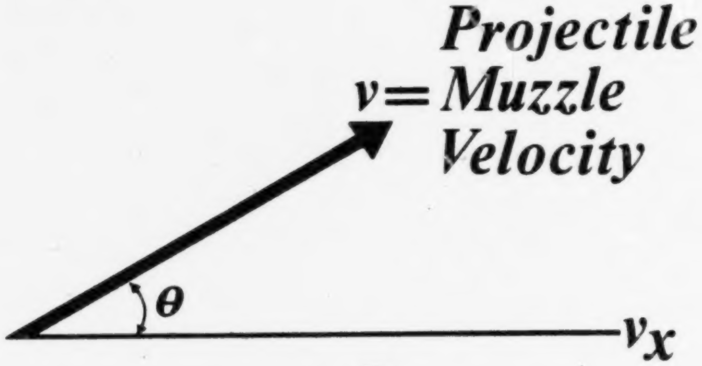

Projectile
v = Muzzle
Velocity
θ
v_x

FIGURE ②

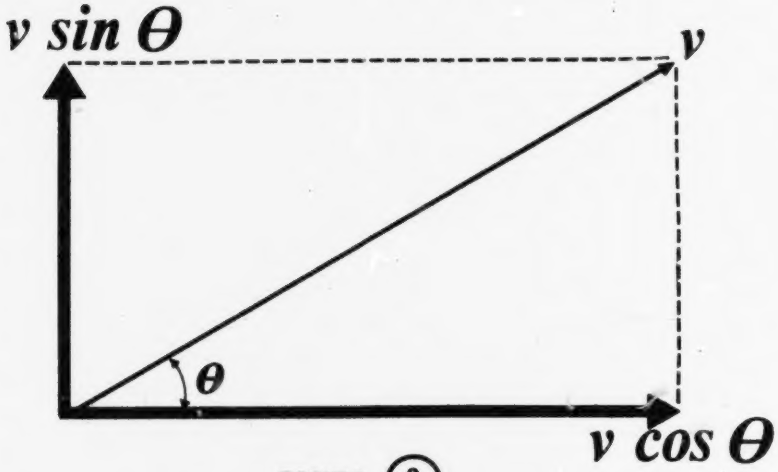

v sin θ
v
θ
v cos θ

FIGURE ③

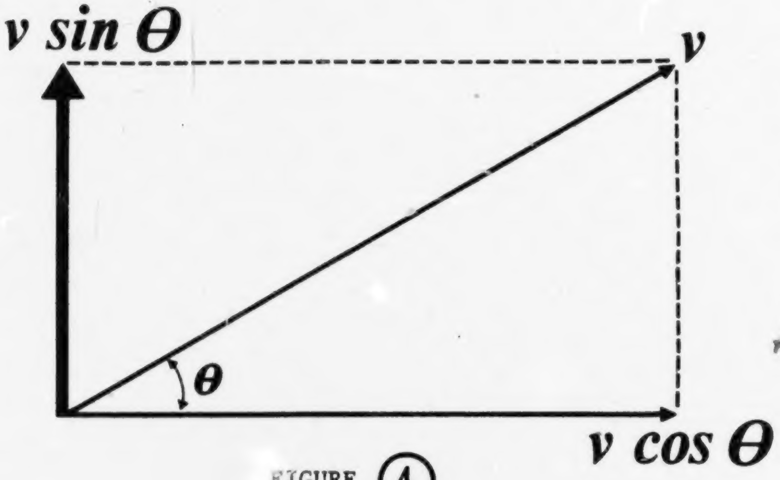

v sin θ
v
θ
v cos θ

FIGURE ④

The real motion of the projectile consists of a vertical and horizontal component, but it will be shown that these two motions are completely independent of one another. The fact that the projectile is moving horizontally at the same time that it is rising or falling does not affect the rise or fall time. This means that the rise time may be calculated by considering the vertical motion only as in Figure 5. The final velocity v attained by a body moving in a gravitational field is given by the expression $v = v_y - gt$ where v_y is the vertical component of the initial velocity, g is the acceleration due to gravity, and t is the time required for the body to reach the velocity v. When a rising projectile reaches the highest point of its flight, it must stop moving upward just before it begins its descent, hence the final velocity at the top of the flight is zero. Substituting zero for v in the equation yields finally the expression given for the time t to the highest point.

Since the projectile is acted upon by the same accelerating force on its way down as it was subjected to on its way up, the time for return may be shown to be exactly the same as the time of rise. Refer to Figure 6. From this it can be seen that the total time of flight is simply twice the rise time.

$$v = v_y - gt$$

But At Maximum Height $v = 0$

So $v_y = gt$ **or** $t = \frac{v_y}{g} = \frac{v \sin \theta}{g}$

FIGURE ⑤

$$\text{time of flight} = \frac{2v \sin \theta}{g}$$

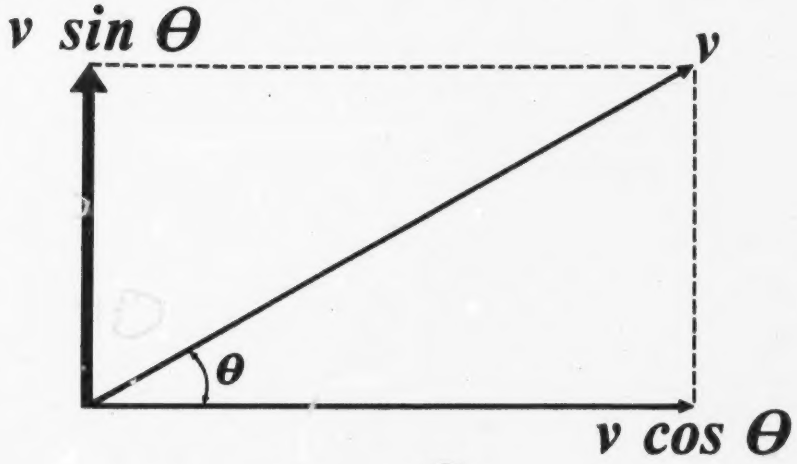

$v \sin \theta$
v
θ
$v \cos \theta$

FIGURE ⑥

Figure 7 also provides an expression for finding the range R of a projectile. The range is defined as the horizontal distance from the cannon to the point where the projectile returns to earth. The range equation is obtained in this way: once the projectile leaves the muzzle, the only force acting on it is the gravitational force provided that we ignore frictional retardation due to air resistance. The gravitational force is wholly vertical; it has no horizontal component. As you will discover later when you study the laws of motion, an object on which no unbalanced force acts neither accelerates nor decelerates. In this case, the absence of an unbalanced horizontal force seems to imply that the horizontal component of the projectile's velocity will be constant. This remains to be seen but it is a justifiable preliminary assumption. Making this assumption, then, it can immediately be said that the distance (range) covered by the projectile is simply:

$$R = v_x t$$

in which R = range, v_x = constant horizontal velocity, and t = total time of flight. It is already known, however, that:

$$v_x = v \cos \theta$$

$$\text{and} \quad t = \frac{2 v \sin \theta}{g}$$

Substituting these identities for the terms in the first equation yields the range equation in Figure 7.

When the range equation is simplified it appears as in Figure 8. This figure presents the two key equations developed thus far in final form. At this point, the student should do a unit check on both expressions to be sure that he sees that, in MKS units, t will come out in seconds and R in meters.

$$t_{flight} = \frac{2v \sin \Theta}{g}$$

$$Range = \frac{2v^2 \sin \Theta}{g} \cdot \cos \Theta$$

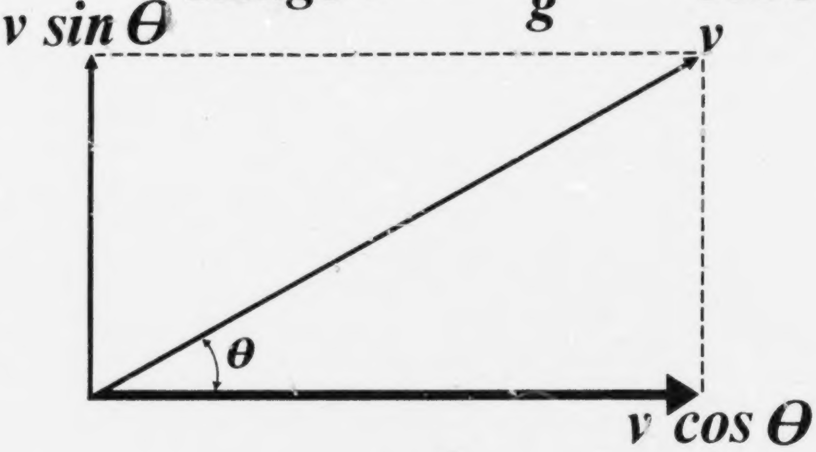


FIGURE ⑦

$$t_f = \frac{2v \sin \Theta}{g}$$

$$R = \frac{2v^2 \sin \Theta \cos \Theta}{g}$$

$$= \frac{v^2 \sin 2\Theta}{g}$$

FIGURE ⑧

It was mentioned above that the absence of an unbalanced horizontal force implies that the horizontal velocity will be constant. This is sometimes demonstrated with the aid of a spring gun (Figure 9) which fires a spherical projectile through the air. A coordinate grid may be used as a background for observing the trajectory (Figure 10).

There are several ways to observe the trajectory so that measurements can be made to confirm the constancy of v_x among other things.

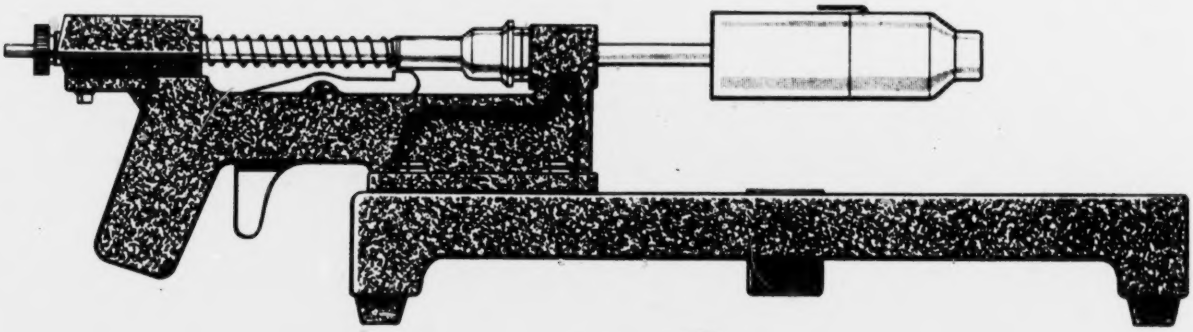


FIGURE ⑨

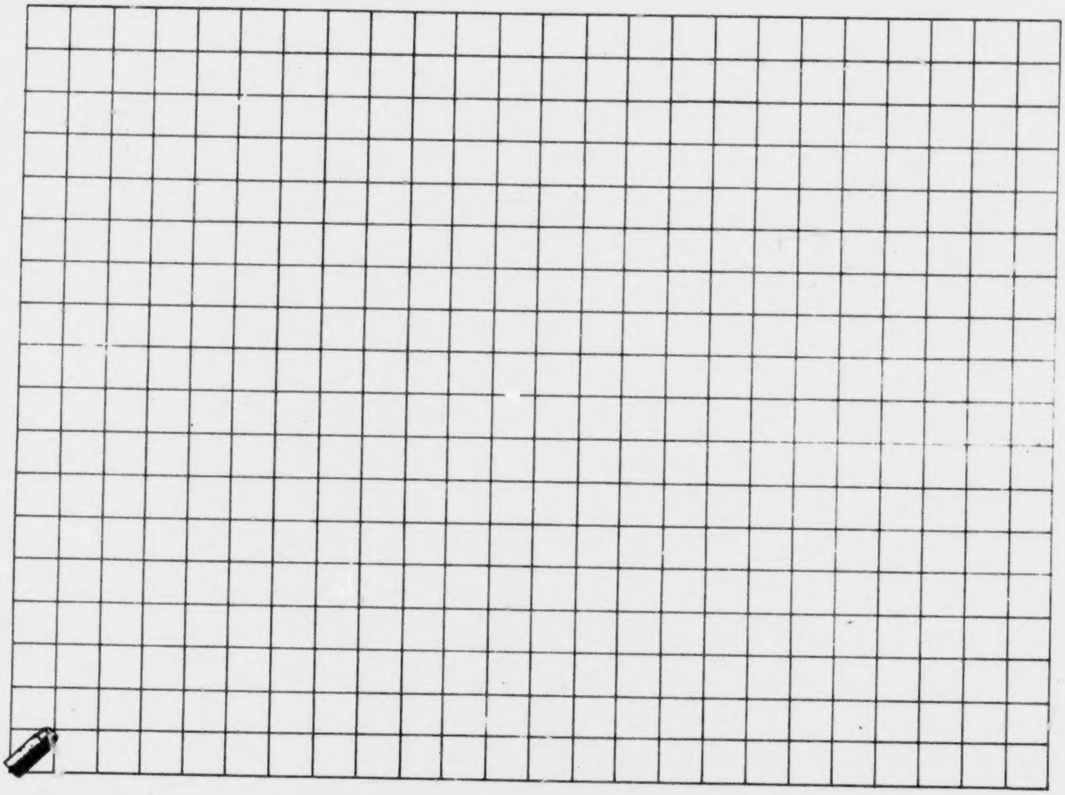


FIGURE ⑩

A high-speed motion picture camera may be used to film the actual flight of the ball and then may be played back in slow motion, or a Polaroid type of camera may be used to produce a composite print if the source of photographic light is a high-intensity stroboscope set for rapid, repetitive firing. Either of these methods presents a final picture such as that given in Figure 11.

Note first that the vertical displacement for equal time intervals constantly changes indicating that the vertical velocity is not uniform. Then observe that the horizontal displacements in each unit of time are the same showing that v_x is constant. The right-left symmetry of the trajectory curve also serves to show that the horizontal motion is uniform; if the projectile were slowing down -- an effect one might expect if a horizontal retarding force were acting on it -- the right-hand portion of the trajectory curve would reveal this in the form of a steepening slope for each unit of time.

It is a matter of interest that the ideal trajectory curve is a parabola that follows the equation:

$$y = ax - bx^2$$

in which y is the vertical height at any time as a function of the horizontal position x, and a and b are constants whose values depend on the angle of elevation of the gun, the initial velocity of the projectile, and the value of the gravitational acceleration constant g at that particular location.

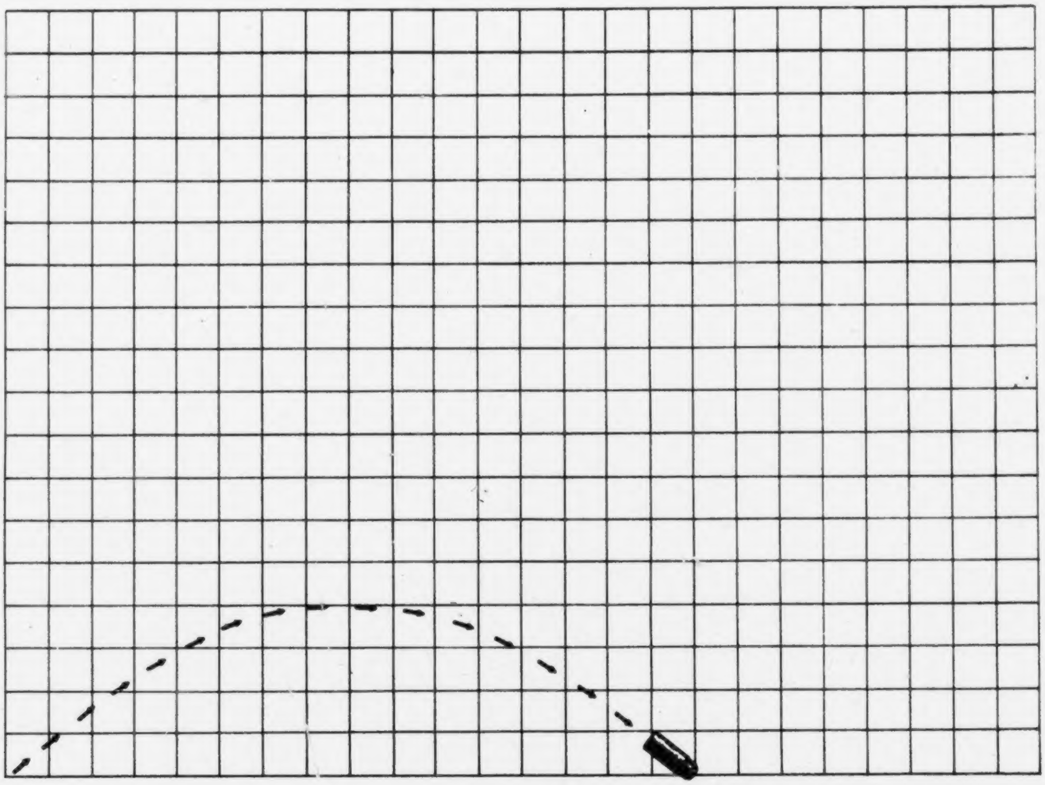


FIGURE ⑪

There is another interesting experiment that may be performed in the classroom. This one is used to demonstrate another important aspect of projectile motion. Refer to Figure 12. Shown here is a "trick" projectile consisting of two halves of exactly equal mass; when assembled, the two parts are held together by a short string which neutralizes the tendency of the internal spring to make the projectile "explode". A carefully timed fuse is set to blow the projectile apart at or near midflight. Suppose, as indicated in Figure 13, the explosion is timed to occur exactly at midflight when the projectile's axis is horizontal. A short time afterward, the two equal-mass fragments would have moved apart to the positions shown in Figure 14. The fragment on the right has gained some additional speed as a result of the explosion while the one on the left lost some speed. The former has flattened its trajectory and the latter shows a steeper trajectory, both the result of the simultaneous changes in speed. But a point of special interest emerges from a study of the positions of the fragments: the midpoint of the line connecting the centers of the two fragments lies on the original trajectory curve.

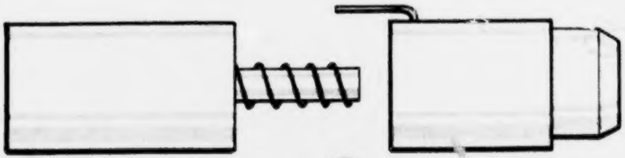


FIGURE ⑫

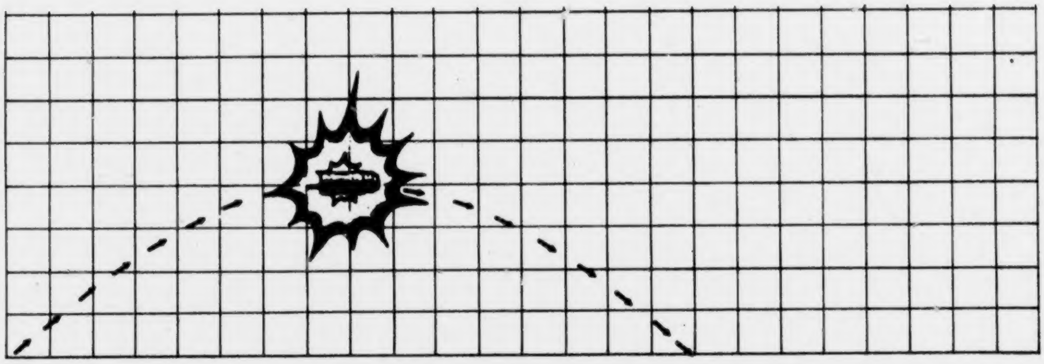


FIGURE ⑬

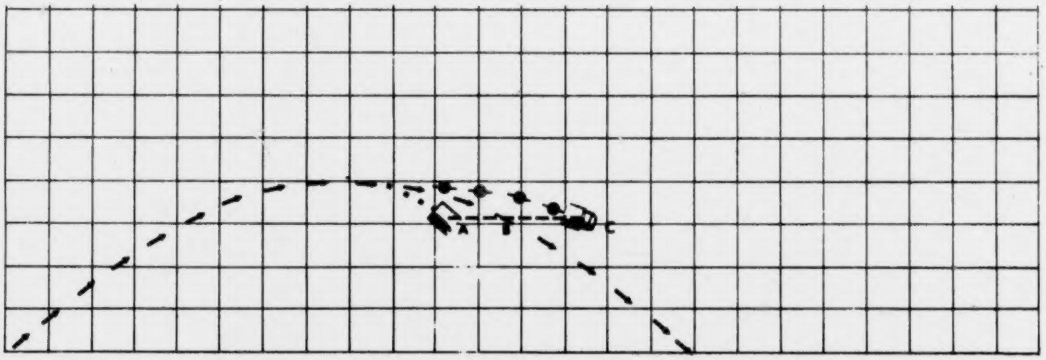


FIGURE ⑭

Now refer to Figure 15 which depicts the fragment positions a little later in time. The trajectories continue to move apart but the center of the connecting line still rests on the original curve.

This action continues right up until the instant of impact of both fragments with the ground, as shown in Figure 16.

Exactly what is the significance of this consistent behavior of the fragments? It points out a very significant phenomenon: the center of mass of the two-fragment system follows the trajectory that the whole projectile would have taken had there been no explosion.

The demonstration described above has two "special-case" aspects: first, the explosion occurred exactly in the middle of the trajectory; second, the axis of the projectile was perfectly horizontal, insuring that no vertical forces would act on it during the explosion. To prove that the motion of the center of mass of the equal fragments would follow exactly the same path for all other conceivable variations requires an understanding of the concepts of momentum and conservation principles. The fact that this does indeed occur is easily shown by experiment but the mathematical proof must be left for a later date.

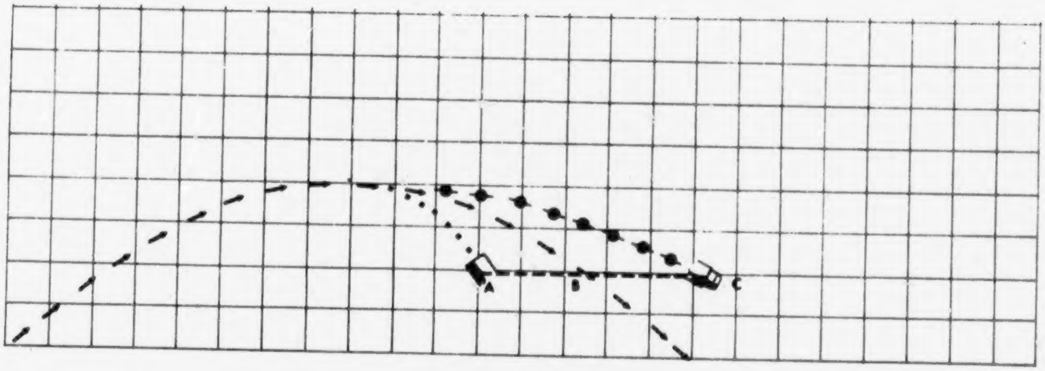


FIGURE ⑮

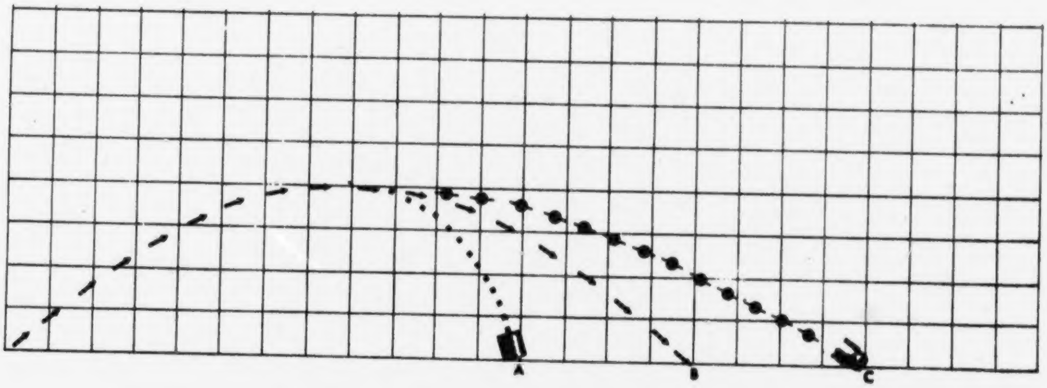


FIGURE ⑯

PROJECTILE MOTION

TERMINAL OBJECTIVES

2/3 B Analyze the trajectory curve of a particle projected horizontally (no vertical component) from the top of a structure.

2/3 E Solve position, time velocity, and range problems involving projectiles with any angle of departure.

Please turn to page 22A of your STUDY GUIDE to continue with your work.

Newton's
1st Law

Aristotelian philosophy held that a material object was in its "natural" state only if it was at rest. If there is no force acting on a body in motion to maintain the state of motion, then the body must come to rest to return to its "natural" state. When one considers that most simple observations made during the normal course of a day seem to bear out this conclusion, one must concede that the assumption appears reasonable. If a massive object such as a well-stocked bookcase were suddenly to rise in the air of its own volition, even a modern observer would consider the action "unnatural" or, more probably, supernatural! If a body at rest does not begin to move unless it is somehow influenced by an external agency, it would appear logical to assume that a moving object would come to rest of its own accord if the agency that caused it to move initially were to be removed. And, indeed, this is precisely what happened in the basic experiments performed by the ancient philosophers. If a book is hand-propelled along a table top and if the hand is then removed, the book comes to rest almost immediately.

The basic fallacy in this reasoning is that one tends to ignore certain external agencies which do not overtly make themselves evident to the senses. When these hidden factors are searched out, exposed, and accounted for -- the "natural" state of things becomes a myth. As a steam locomotive drawing a train of cars stoutly puffs and snorts, it certainly appears as though the force exerted by the engine on the wheels is needed to keep the train moving at a constant speed. But there are "hidden" forces acting on the train. One of these is illustrated in Figure 1. It is the retarding force offered to the motion of the locomotive by the air itself. At any reasonable speed, the locomotive must push its way through the enveloping atmosphere and as it does so, must thrust the air out of its path. The air returns the thrust in the form of an opposing force which, at high speeds, becomes quite large.

A second opposing force takes the form shown in Figure 2. There is friction between the axle bearings and the wheels; there is friction between the wheels and the track despite the rolling action. Thus friction is the second retarding force that must be overcome if the train is to move.

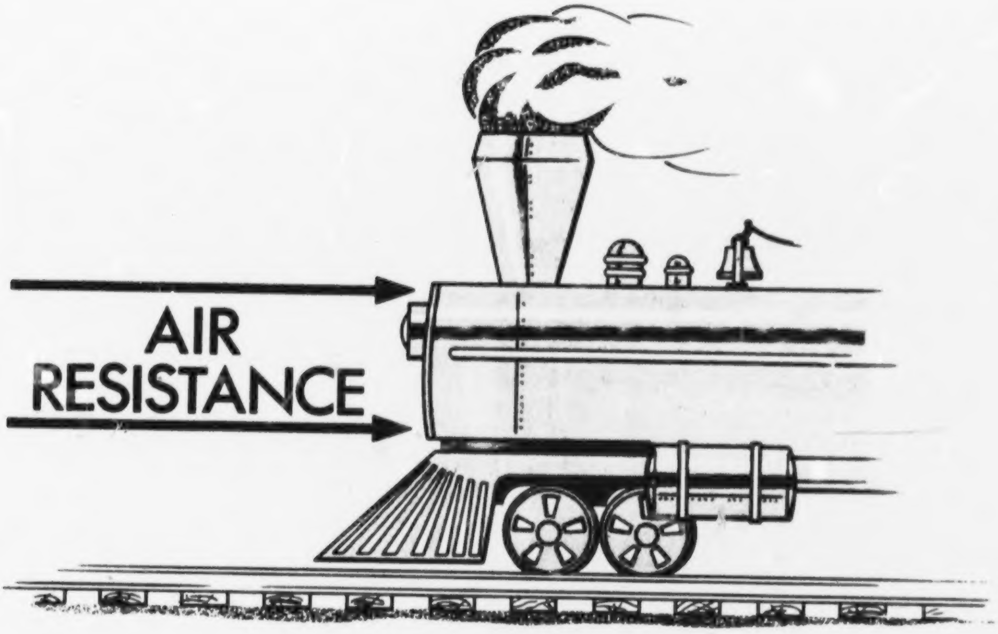

AIR
RESISTANCE

FIGURE ①

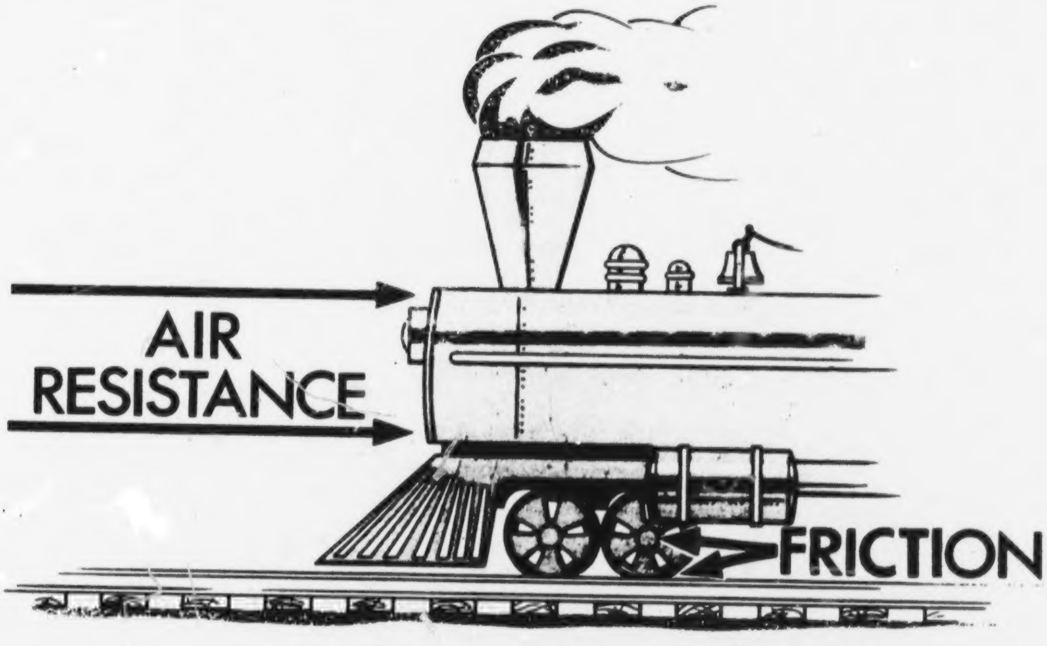

AIR
RESISTANCE
FRICTION

FIGURE ②

Careful measurement of the pull of the locomotive engine and the magnitudes of the two retarding forces just described discloses that the sum of the opposing forces is equal to the force exerted by the engine on the wheels. This is graphically illustrated in Figure 3. Note that the engine thrust is directed oppositely to the sum of the retarding forces. Hence, the net or unbalanced force acting on the train is zero. This leads to the conclusion that the train will continue to move with unchanging speed as long as there is no unbalanced force exerted on it. Thus, it is apparent that the ancient belief regarding the naturalness of the rest state is incorrect. A state of uniform motion -- unchanging velocity along a straight path -- is just as natural.

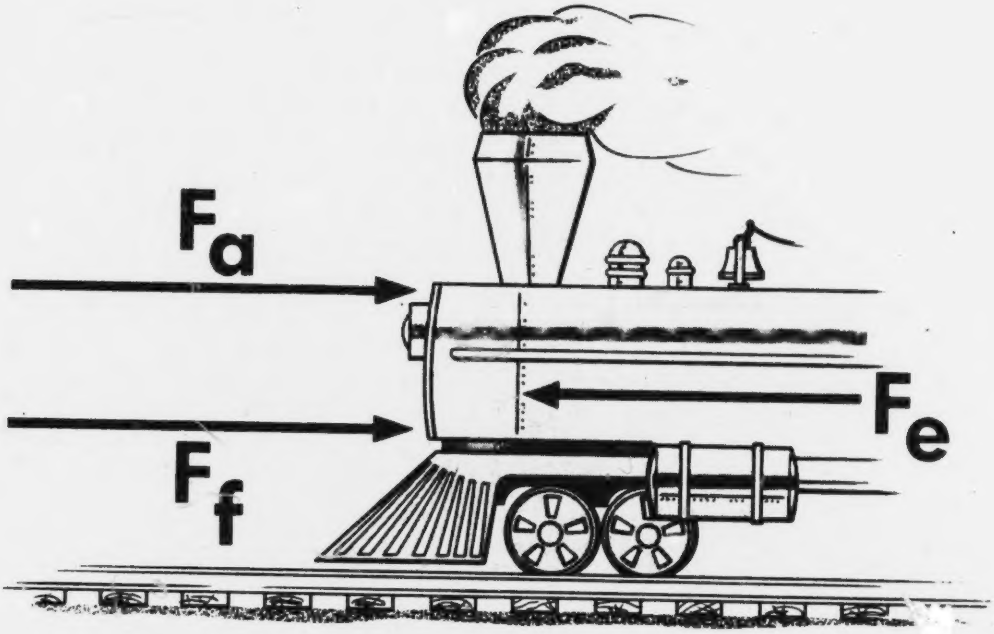


FIGURE ③

Although the Italian scientist Galileo recognized the error of the ancient logic and proposed a number of "thought" experiments to show that the conclusions derived therefrom were untenable, it was Isaac Newton who formalized the generalization which correctly links these concepts. In his Principia, a written work that still is regarded by many as one of the most outstanding scientific documents of all time, he states (in Latin) that "every body continues in its state of rest, or of uniform motion in a straight line unless it is compelled to change that state for forces impressed on it." This statement, in somewhat more modern garb, is presented in Figure 4.

You have probably seen some of the films taken inside space vehicles by U.S. astronauts on their way to or from the moon. In these views, you have been treated to experiences that the ancients could not possibly have enjoyed. A wrench remains floating in the cabin when the astronaut removes his hand. From the point of view of the astronaut -- as seen by his camera -- the wrench is at rest. The outside observer, however, is aware that this is not true from his viewpoint; he sees the wrench moving with uniform speed, keeping pace with the vehicle as it progresses along a straight line between the earth and the moon. Particularly when the vehicle reaches the gravity-null point between the two bodies, where gravitational effects may be completely ignored, the wrench is a body on which the net or unbalanced force is really zero yet it continues to move with unchanging velocity.

A number of important implications of the first law are given in the figures that follow. The statement in Figure 5 also implies that zero resultant force is the equivalent of no force at all.

Figure 6 defines by implication the so-called inertial frame of reference. The floating wrench appears stationary to the astronaut but appears to be moving with uniform velocity as seen by the outside observer. This means that the concepts of "absolute" motion and "absolute" rest are quite meaningless. All motion is relative; motion can be defined only by referring to a preselected set of coordinates.

Newton's First Law embodies the true concept of a "force". Refer to Figure 7. Forces do not give rise to or maintain uniform motion; they bring about changes in motion. When a body at rest relative to a given observer begins to move, he must conclude that a force is acting on the body in the direction of the observed motion. When a moving body is observed to slow down, he must conclude that a force opposite to its direction of motion acts on it. And, finally when a body is observed to follow a curved path, he must conclude that a force having a component perpendicular to the line of flight must be acting on the body to cause this deviation from a straight path.

NEWTON'S FIRST LAW
OF MOTION

A BODY REMAINS AT REST OR IN MOTION WITH UNIFORM VELOCITY UNLESS ACTED UPON BY AN EXTERNAL, UNBALANCED FORCE

FIGURE ④

ONCE A BODY HAS BEEN SET IN MOTION IT IS NO LONGER NECESSARY TO EXERT A FORCE ON IT TO KEEP IT MOVING.

FIGURE ⑤

THE MOTION OF AN OBJECT CANNOT BE SPECIFIED UNLESS THIS MOTION CAN BE REFERRED TO SOME OTHER BODY.

FIGURE ⑥

FORCE IS THAT WHICH CHANGES THE STATE OF MOTION OF A BODY.

FIGURE ⑦

Newton's 1st Law

TERMINAL OBJECTIVES

3/2 A Analyze and interpret a variety of natural phenomena relevant to Newton's First Law of Motion in terms of the First Law.

Please turn now to page 15A of your STUDY GUIDE to continue with your work.

Newton's 2nd Law

Newton's First Law is specifically concerned with bodies that are either at rest relative to the observer, or in motion with uniform speed in a straight line. The first law emphasizes that a body will remain at rest if it is motionless to begin with, or that it will maintain uniform motion if it is initially moving, unless some outside agency capable of exerting an unbalanced force acts upon it.

As might be anticipated, Newton's Second Law describes the relationships among the factors that influence a body while it is changing speed or direction of motion. A body at rest or in motion with constant speed in a straight line is not accelerating; the moment acceleration enters the picture, the first law no longer applies.

(Figure 1) Consider an ordinary simple pendulum swinging back and forth on a frictionless bearing. Throughout a single swing, say from B to C in the drawing, the velocity of the bob changes from zero at B and C to the maximum speed it can have at point A. Since the bob must come to rest before reversing direction, points B and C must be places where the velocity is zero; through the distance from B to A, it must pick up speed, reaching maximum at A and slowing down thereafter until it rises to point C.

The first law states that the unbalanced force on the body is zero if the velocity of the body is constant. In this sense, the first law defines force as a physical quantity needed to change the velocity of an object. Since the velocity of the pendulum bob varies continuously throughout its motion, some kind of unbalanced force must be acting on it at all times. (Strictly speaking, there is one point in the swing of a pendulum where the unbalanced force in the direction of motion is zero and the velocity constant. This point lies at the lowest point in the swing.) Newton's interest lay in the relationship he was certain existed between the force applied to a given body and the acceleration it would acquire as a result of this force.

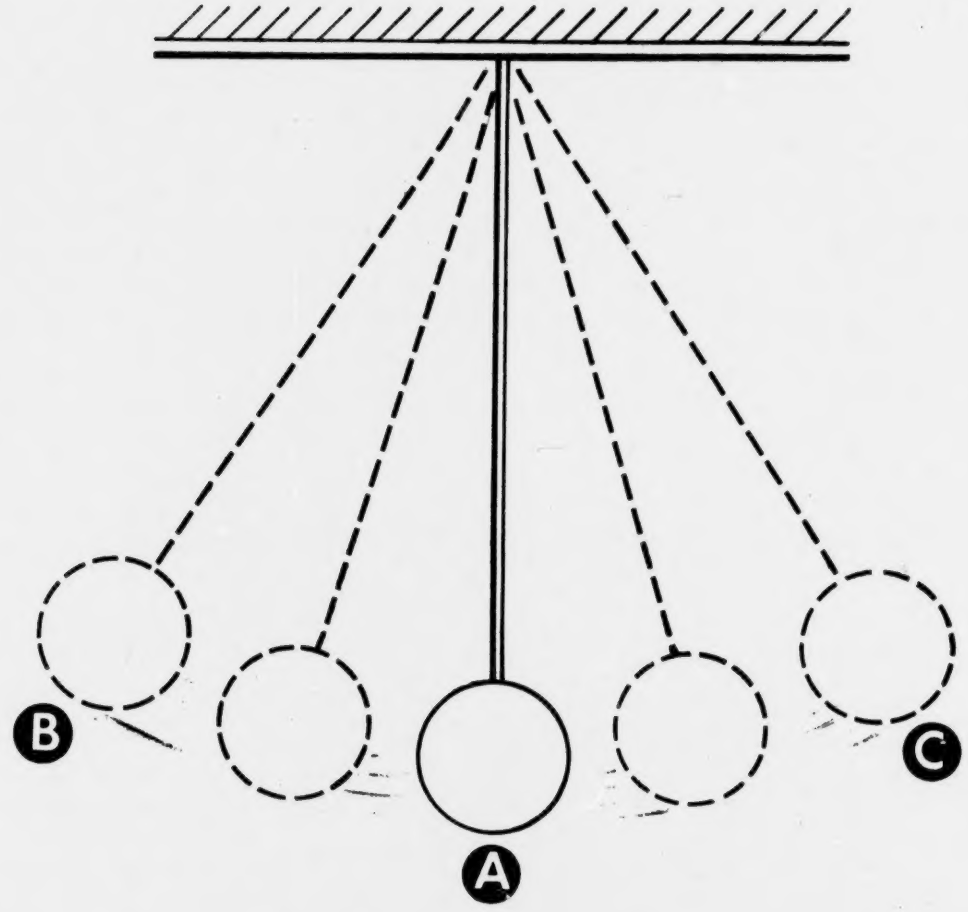


FIGURE ①

(Figure 2) With the intuition of sheer genius, Newton was able to phrase the relationship in the very simple terms shown in this figure. The first statement is a vector equation which indicates that the acceleration of a body is directly proportional to the unbalanced force in the direction of the force. Clearly, mass in this relationship is a constant of proportionality. If the direction of the unbalanced force is constrained along, say, the x-axis of a set of coordinates (F_x), then the acceleration will occur along the x-axis, too, so that the equation may be written in scalar form as shown in the lower expression. Alternatively, the scalar equation may be used when the unbalanced force is applied in the same direction as the body is already moving (or in the opposite direction) since the vector signs are unnecessary in this special case.

The relation $F_x=ma_x$ implies, then, that if the force is doubled, the acceleration will double (not the velocity); if the force is reduced to 1/3 of its initial value, the acceleration will go down to 1/3 of its former value. In all cases, the mass is assumed to remain constant.

There are many ways to demonstrate the validity of Newton's Second Law with real moving objects. One rather ingenious method involves a pendulum carried by an accelerating body.

NEWTON'S 2nd LAW

$$\vec{\mathbf{F}} = m\vec{a}$$

$$F = m a_x$$

FIGURE ②

(Figure 3) This can be done with a toy car driven by a small fuel-burning jet engine. The pendulum pivot is mounted on a small mast secured to the car. When the jet engine is turned on, the car accelerates carrying the pivot with it. However, since the horizontal force is not immediately applied to the bob of the pendulum, it tends to stay behind until it is accelerated from rest by the pull of the slanted string. Behind the string is a scale marked off in such a way that the extent to which the string slants backward can be read off.

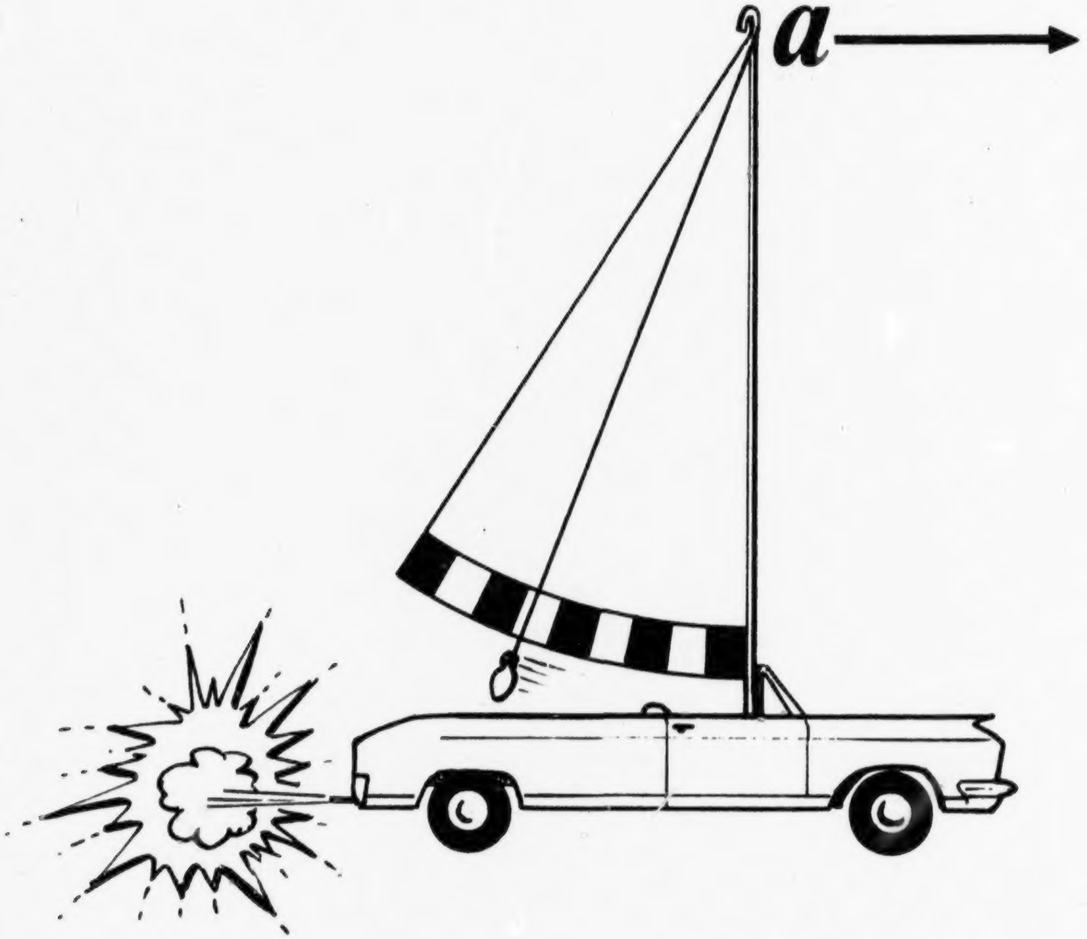


FIGURE ③

(Figure 4) To analyze the motion of a pendulum, it is best to start with the bob hanging straight down as it does when the pendulum is at the bottom of its swing or when it is motionless as shown. It does reside in the earth's gravitational field so that it is subject to gravitational acceleration g. This is the downward acceleration the bob would have if the string were cut so that it could not provide the force that balances gravitation.

(Figure 5) Now imagine that the pivot of the pendulum is given an acceleration a to the right along the horizontal or x-axis. As mentioned previously, the bob will trail behind until the string slants enough to produce some angle θ with the vertical line dropped from the pivot. As will be shown later, as long as the acceleration imparted to the pivot remains constant, the angle of slant θ will also remain constant.

FIGURE ④

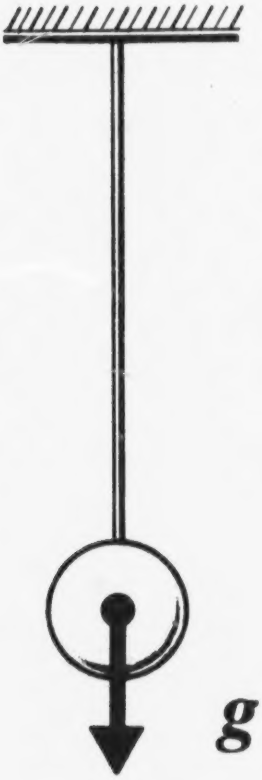
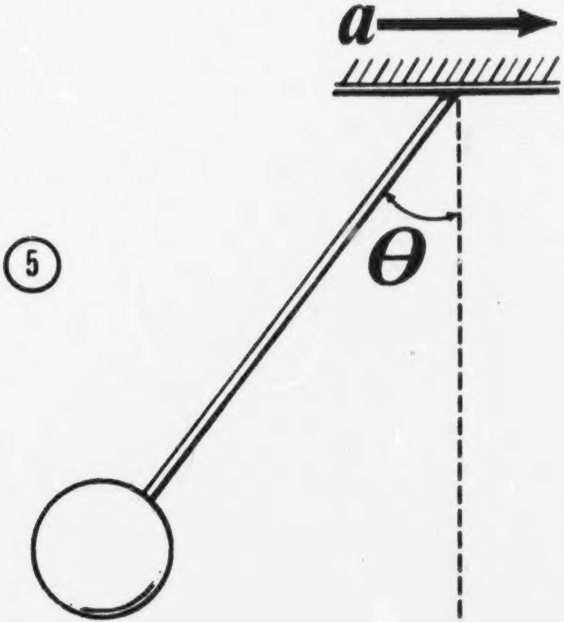


FIGURE ⑤



(Figure 6) In this figure, the bob is shown to have an acceleration a in a direction opposite that of the pivot. To see why this is done, consider the motion when the pivot first begins to move. The bob, with zero horizontal force acting on it at this time, remains where it is on the x-axis. This means that it is accelerating backward relative to a fixed y-axis at the same rate that the pivot is accelerating forward relative to the same axis. At the instant shown, the bob is subject to two accelerations: g, the downward acceleration due to gravity, and a, the relative acceleration of the bob which, as has been shown, is equal in magnitude to the actual acceleration of the pivot.

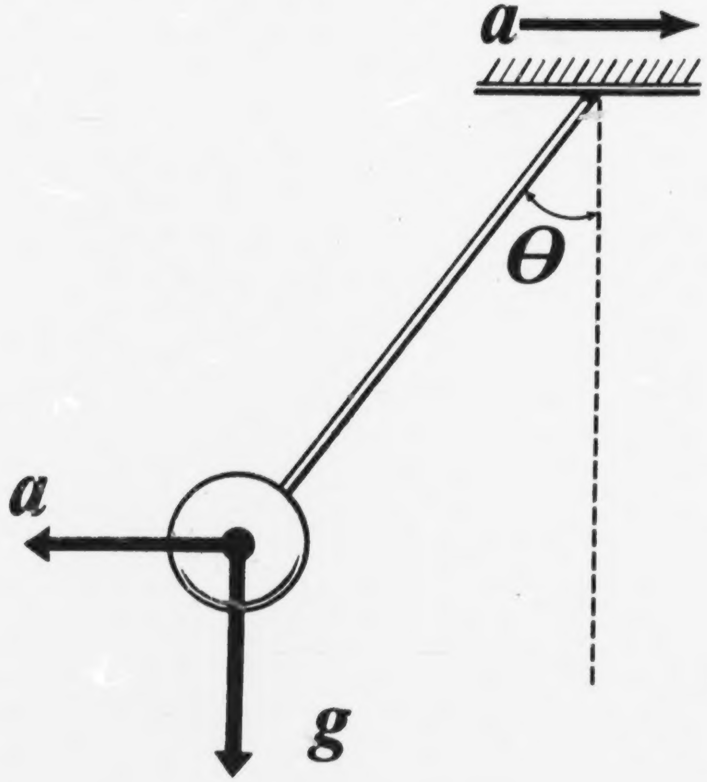
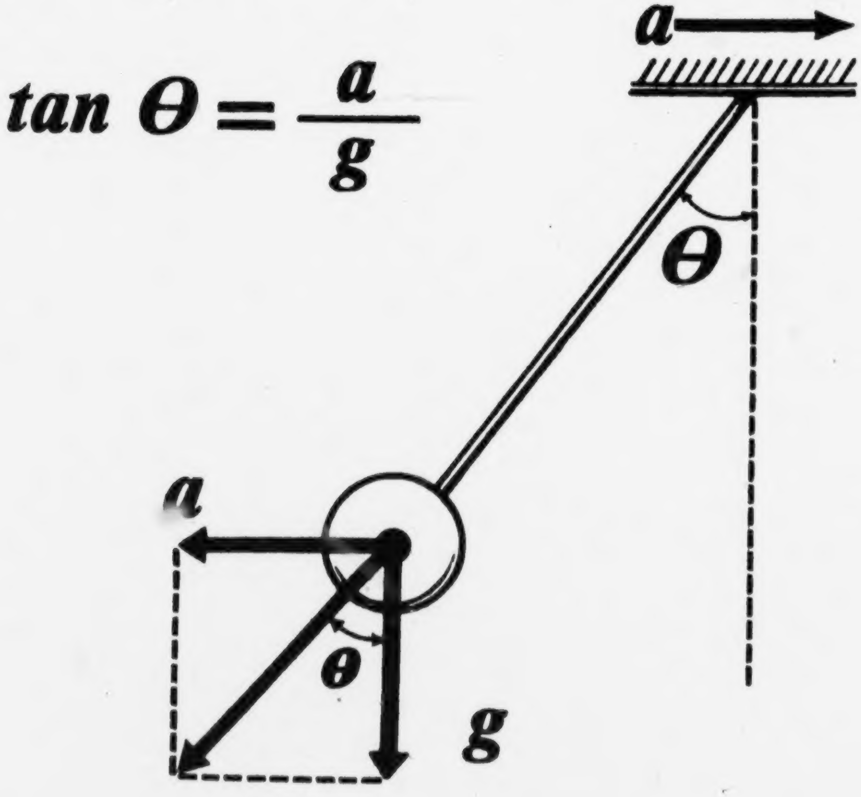


FIGURE ⑥

(Figure 7) In this drawing, the slant angle θ has been brought down into the vector parallelogram. It is immediately obvious that the tangent of the angle is a/g, or $a = g \tan \theta$. The scale behind the string in the toy apparatus previously described may thus be calibrated directly in terms of acceleration. This is the procedure when the experiment is actually performed.

In doing this experiment, certain reasonable assumptions must be made. First, it is assumed that the jet engine provides uniform force throughout the short displacement of the car over the interval of observation. According to Newton's Second Law ($\vec{F} = ma_x$), the acceleration should also be constant throughout the trip under conditions of constant force because mass is assumed to remain constant. This is the second assumption; it is quite valid for velocities that do not approach that of light. Also, the small amount of fuel used during the short trip is taken as negligible. With constant acceleration, the slant angle also remains constant throughout the motion. If the car is brought to an abrupt halt by some obstacle at the finish line, the bob will swing over an equal angle in the forward direction making it rather easy to read θ, or the actual magnitude of the acceleration from the calibrated scale.

The entire experiment just described is performed for the purpose of determining the acceleration of the car in an easily observable manner. The remaining two quantities, the force F_x and the mass $\underline{m}$, are readily measured. The force is obtained by connecting a spring balance between the car and a rigid support along a horizontal line; the jet engine is then fired up as before and the force read directly from the balance. An equal arm balance provides the means for measuring the mass directly.



$$\tan \theta = \frac{a}{g}$$

FIGURE ⑦

(Figure 8) The values given in the figure were actually obtained when this experiment was performed. An unbalanced force of 0.15 newtons was measured on the balance. When this force was applied to the car, the slant angle indicated an acceleration of 1.5 m/sec^2. The mass of the car and engine turned out to be very nearly 100 grams or 0.10 kg. Thus, even in a crude type of measurement such as this it is evident that the product of the mass in kilograms and the acceleration in meters per second per second is indeed equal to the unbalanced force in newtons.

Despite its apparent simplicity, Newton's Second Law still stands as one of the greatest triumphs of a great physical scientist -- perhaps the greatest of all time -- Sir Isaac Newton.

$$a = 1.5\ m/sec^2$$

$$F = 0.15\ nt$$

$$m = 0.1\ kg$$

FIGURE ⑧

Newton's 2nd Law

TERMINAL OBJECTIVES

3/2 B Analyze and interpret a variety of natural phenomena relevant to Newton's Second Law in terms of the Second Law.

Please turn to page 23A of your STUDY GUIDE to continue with your work.

Newton's 3rd Law

This brief discussion of Newton's law of motion is to be based upon a simulated experiment that can be readily duplicated with extremely simple equipment.

The third law has been stated and restated in a multitude of forms. For the purpose of this discussion, the form given in Figure 1 will be utilized.

When one analyzes this statement, it is apparent that it implies the following:

1. A force cannot exist alone; forces always come in pairs;
2. Two bodies are involved in the application of any force;
3. A force applied by one body, say body A, may be called an action. The body on which the "action" acts is another body -- body B. Body B then applies an equal force oppositely directed on body A; this force is designated the "reaction".

The alternative statement; "For every action there is an equal and opposite reaction" is acceptable only if one mentally adds the fact that "action" applies to one body while "reaction" applies to a second body.

IF BODY A EXERTS A FORCE ON BODY B, THEN BODY B EXERTS A FORCE OF EQUAL MAGNITUDE, OPPOSITELY DIRECTED, ON BODY A

FIGURE ①

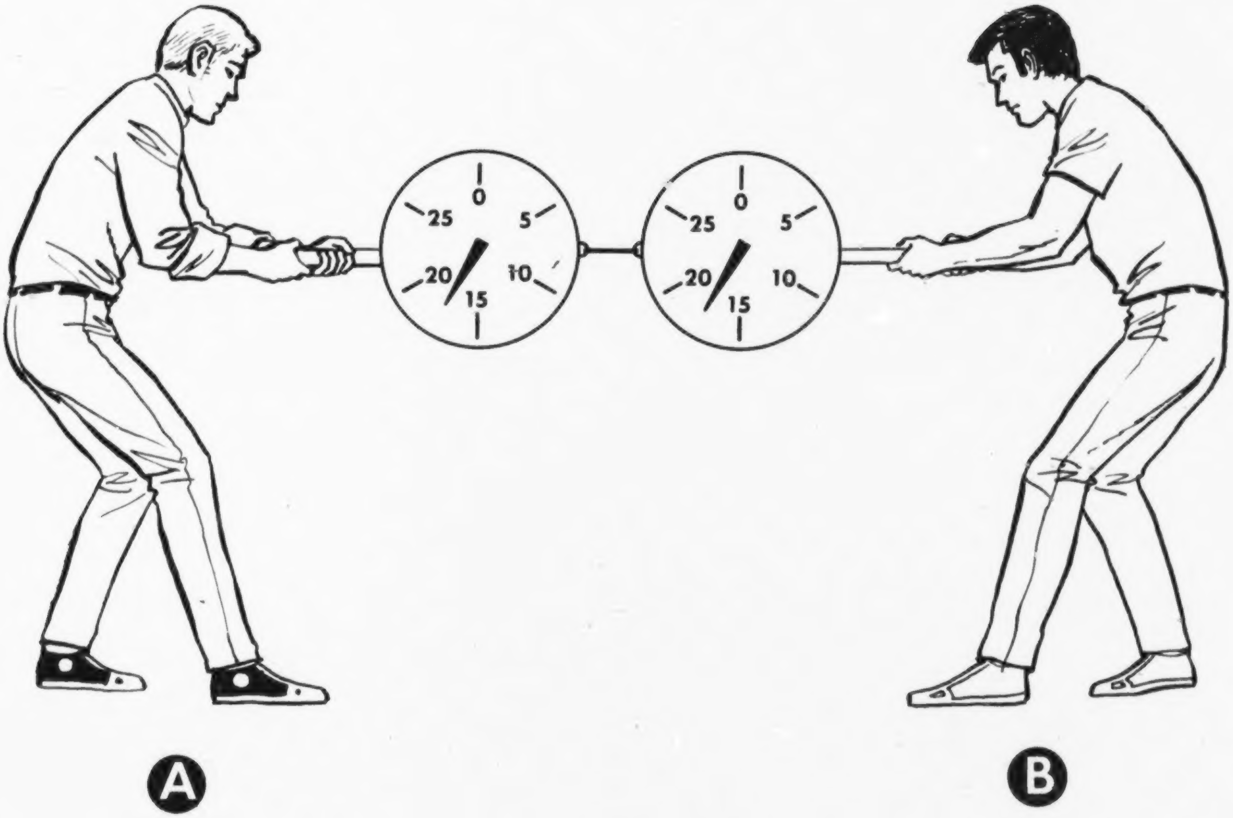
Newton's third law may be expressed symbolically as shown in Figure 2. In this statement F represents the force considered to be the "action" and R represents the "reaction". The presence of the negative sign before the "R" specifies the oppositeness of the reaction force.

NEWTON'S THIRD LAW OF MOTION

$$F = -R$$

FIGURE ②

The simulated experiment begins with two people, A and B, and a spring scale which reads up to about 30 units of force. The actual unit used is of no consequence. A holds the ring of the balance and B proposes to exert a force on the hook thereby causing the balance to indicate the magnitude of the force, as shown in Figure 3.



0
5
10
15
20
25
0
5
10
15
20
25
A
B

FIGURE ③

A says to B, "Let's see you exert a force of 15 units on the hook so that the scale dial will read that figure." As B starts to pull on the hook, A begins to move toward B -- in the same direction as the force B is trying to exert -- thus giving way to B's pull by matching his attempt to pull on the hook as in Figure 4.

With the hook moving toward him as fast as he pulls it, B finds that he cannot make the balance giving any reading other than zero. Since A permitted the balance to move toward B, there was no reaction force against which B could apply his force.

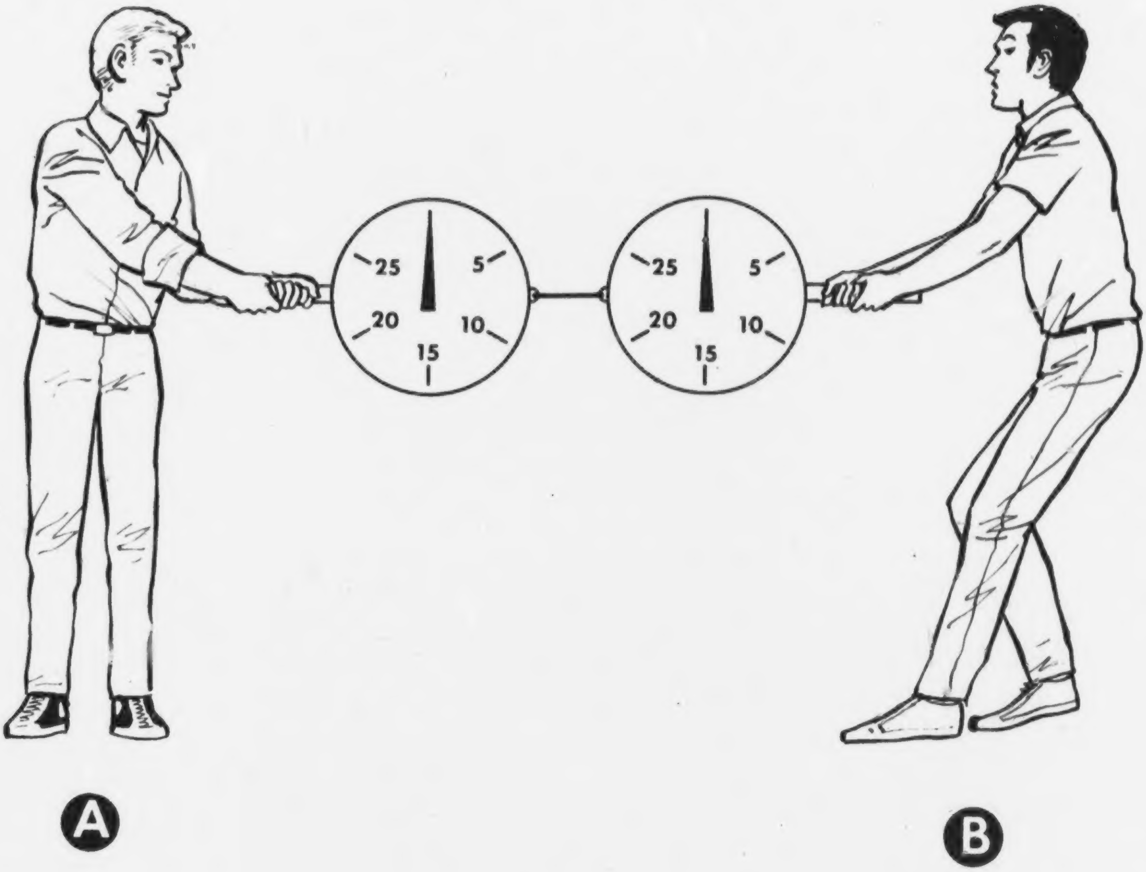

25
5
20
10
15
25
5
20
10
15
A
B

FIGURE ④

On the other hand, if A holds the ring of the balance firmly and does not give ground, there will be a reaction against which B can exert his force as illustrated in Figure 5. In this case, as long as A does not permit the balance to move with B's pull, B can make it read anything he likes within the capabilities of his physical strength. Note that A really does exert a force to the left to hold the scale motionless while the force exerted by B can stretch the spring and cause the needle to rotate on the dial.

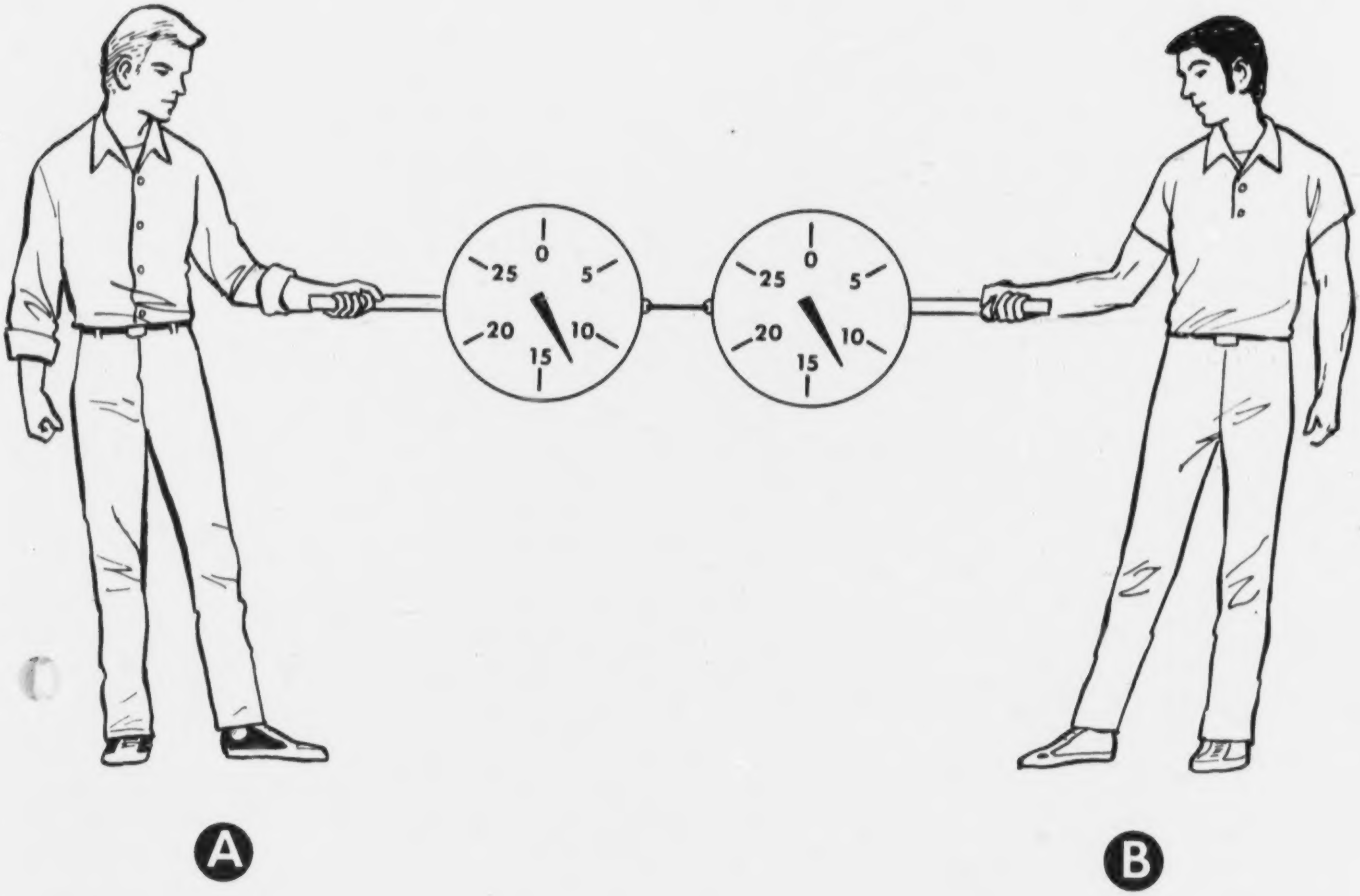

0
5
10
15
20
25
0
5
10
15
20
25
A
B

FIGURE ⑤

The experiment thus far has demonstrated the need for force pairs in nature. The next part of it proves that these oppositely directed forces are indeed equal in magnitude. Referring to Figure 6, two identical balances are illustrated, one held by A and the other by B. At the instant shown in the diagram, neither person is exerting a force, hence both balances read zero. Next, each of the participants is told to exert a specific force on the hook he holds in his hand: A is told to make his balance read 5 units while B is instructed to cause his to read 15 units. The result? No matter how earnestly each of the people tries, he cannot follow his instructions. Regardless of the disparity in weight or size of the participants, they cannot bring about the scale readings desired.

The actual result is shown in Figure 7. Both balances give identical readings at all times; they quiver, oscillate, waver, and jump around as the participants tug and give way, but their needles remain in exact synchronization throughout. If B pulls harder, his balance reading rises but so does A's; if either one relaxes his pull, both readings go down equally.

Forces exerted this way form an action-reaction pair; at any given instant, the two forces must be equal in magnitude but opposite in direction.

A simple experiment like this is most convincing. In particular, it shows that forces do indeed come in pairs and that two bodies are always involved. The force that A exerts on B must be equal in magnitude and opposite in direction to the force exerted by B on A. That is, $F = -R$.

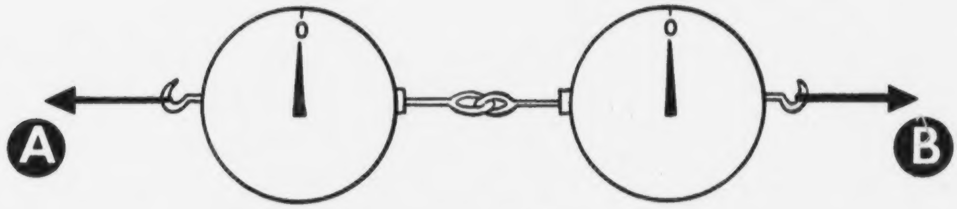


FIGURE ⑥

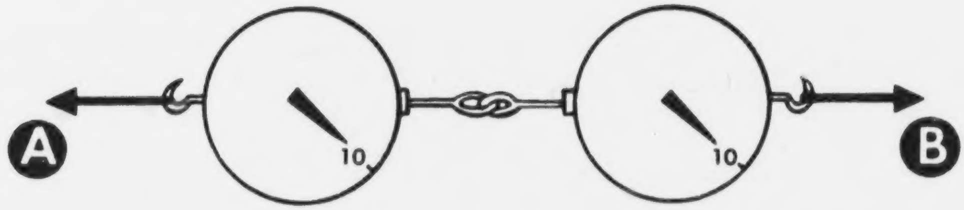


FIGURE ⑦

Newton's 3rd Law

TERMINAL OBJECTIVES

3/2 C Analyze and interpret a variety of natural phenomena relevant to Newton's Third Law of Motion in terms of the Third Law.

Please turn to page 35A of your STUDY GUIDE to continue with your work.

I

ATWOOD'S MACHINE

The acceleration of a freely-falling body on the surface of the earth is roughly 32 ft/sec/sec or 9.8 m/sec/sec. When an object is allowed to fall freely over even the longest distances normally available in the physics laboratory, the time of fall is too short to permit measurement with any degree of precision using a standard stopwatch. This makes the direct measurement of g difficult unless special measuring devices are available.

The Atwood's machine overcomes this difficulty. Essentially, the machine is designed to dilute gravity by a known or readily calculable factor; the acceleration of a falling mass is then measured with standard tools and g calculated with the help of a simple equation which will be derived in this discussion. The usefulness of the Atwood's machine may also be extended to a study of the forces that govern the behavior of the string-mass-pulley system typical of this machine.

The original Atwood's machine is shown in Figure 1. It consists of a single pulley, a string, and a pair of masses, either one of which may be individually changed. The double-pulley arrangement illustrated in Figure 2 is a laboratory modification of the original; it is merely somewhat more convenient to use but it changes nothing of the Atwood concept.

Two fundamental assumptions are required to idealize the laboratory equipment: (1) the pulleys are frictionless; fine ball-bearing pulleys are available so that this assumption is very closely approximated; (2) the string is massless and inextensible. The use of a special nylon string makes the actual situation approach the ideal satisfactorily.

Suppose that the two masses in Figure 2 are equal. For this condition, the system will remain in equilibrium no matter where the masses are placed. Since the value of g for each mass may be taken to be the same, Newton's Laws may be readily applied to explain this result. Consider the free-body diagram of either mass shown in Figure 3. The weight of this mass acts downward from the center of gravity as indicated by the vector arrow pointing downward. A second vector arrow pointing upward represents the tension (force) exerted by the string on the mass. Its length is equal to that of the weight vector to point out that the two forces are equal in magnitude but oppositely directed. The resultant vertical force is then zero and the system remains in equilibrium.

Consider now that a small additional mass is added on one side as in Figure 4. When the string is released, m_1 accelerates downward while m_2 accelerates upward. Because the same string is attached to both masses, it is justifiable to assume that the same tension exists throughout the string. (If the string had mass, this assumption would not be strictly correct but in this idealized situation it is quite accurate.)

ATWOOD'S ORIGINAL MACHINE

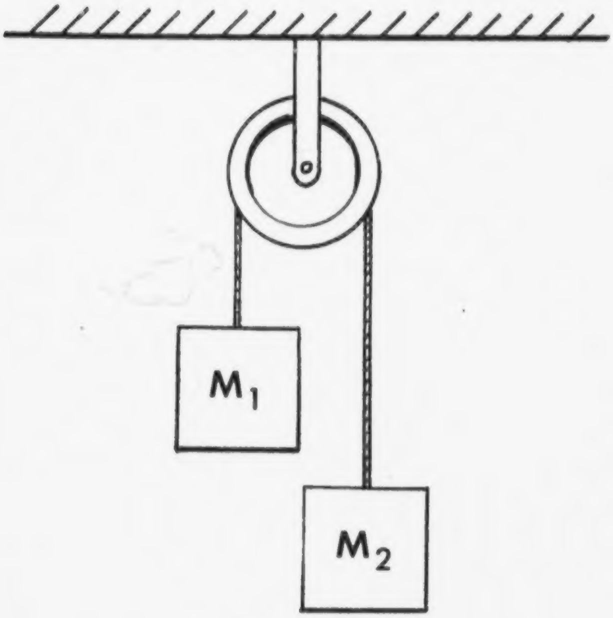


FIGURE ①

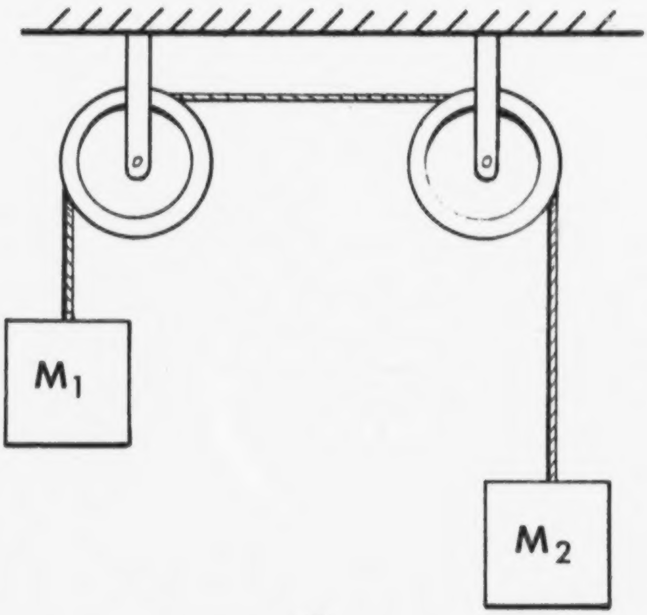


FIGURE ②

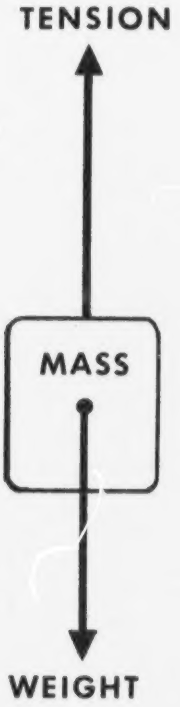

TENSION
MASS
WEIGHT

FIGURE ③

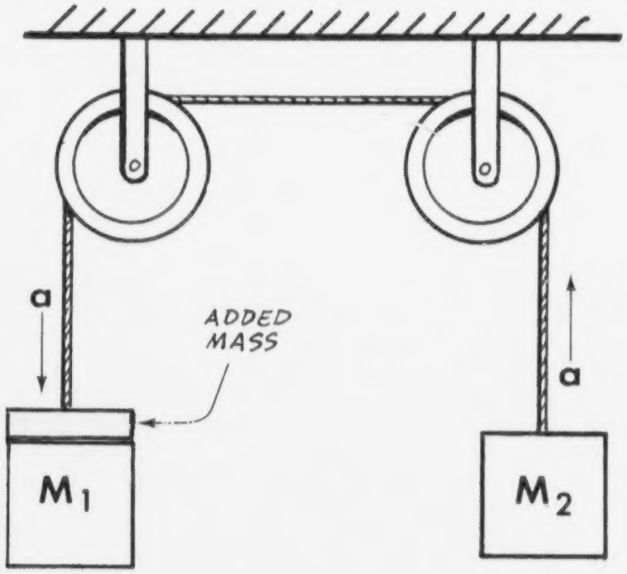
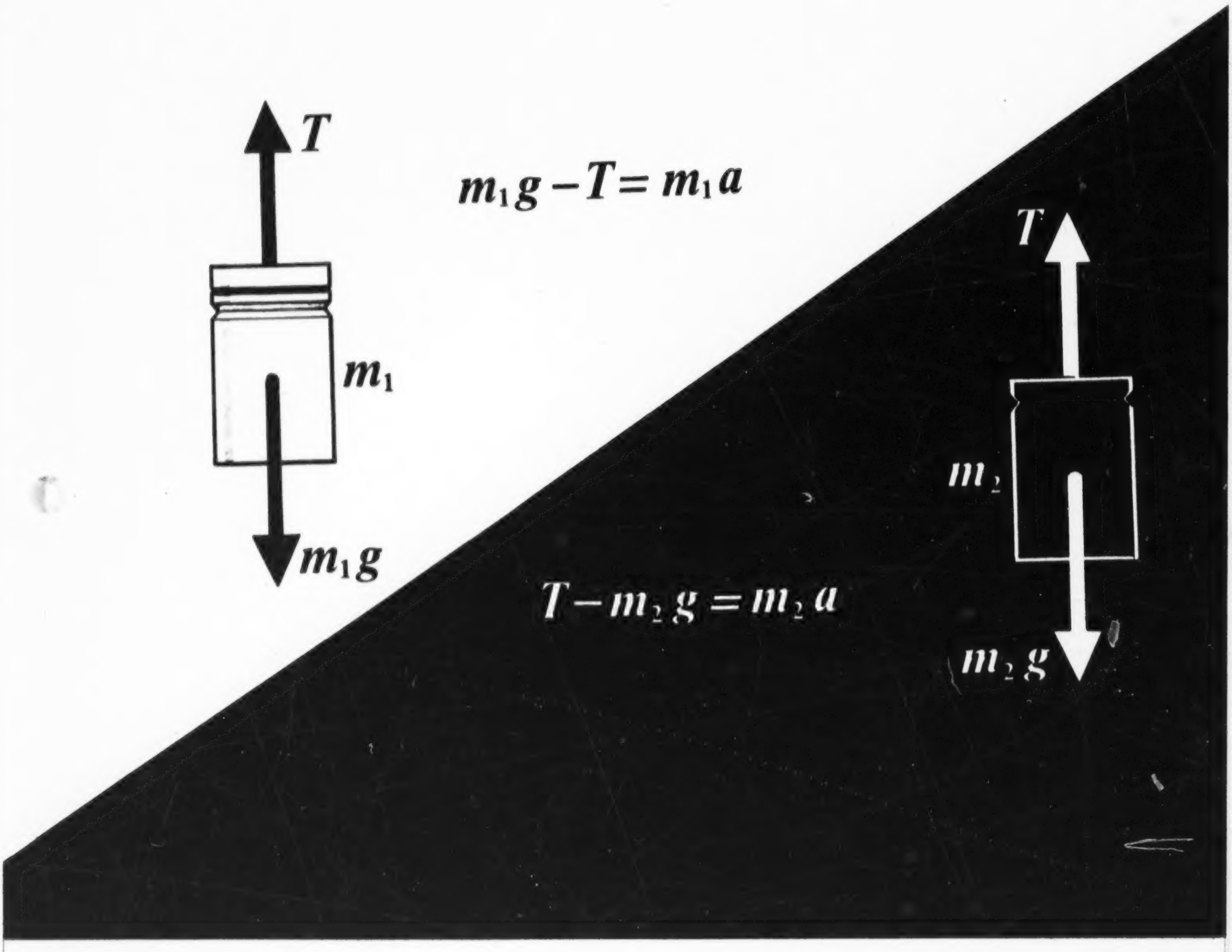

a
ADDED MASS
a
M$_1$
M$_2$

FIGURE ④

The free-body diagrams for both masses during the acceleration process are given in Figure 5. Additional mass has been placed on the left making m_1 larger than m_2. Thus, the weight of m_1, that is, m_1g is greater than the tension. The difference between m_1g and the tension T represents an unbalanced force acting downward on this mass so that it accelerates in this direction. The magnitude of the acceleration is, of course, given by the second law -- $F = ma$ -- and is shown as "a" in Figure 4. The unbalanced force is $m_1g - T$ and may be substituted for F in the second law equation yielding: $m_1g - T = m_1a$. The mass on the right accelerates upward at the same rate -- again because the string is massless and inextensible. In this case, T is larger than m_2g. The second law equation for m_2 is, therefore, $T - m_2g = m_2a$. These equations should be studied carefully before proceeding since they are key statements.



T
$m_1 g - T = m_1 a$
m_1
$m_1 g$
T
m_2
$T - m_2 g = m_2 a$
$m_2 g$

FIGURE ⑤

Clearly, T can be eliminated from the equations by adding them algebraically. This procedure is demonstrated in Figure 6. The final equation:

$$a = \frac{m_1 - m_2}{m_1 + m_2} \cdot g$$

provides the "dilution factor" that makes the Atwood's machine so useful. This factor is the fractional term on the right side. It should be noted that it will be a small proper fraction if m_1 is not made much larger than m_2. In other words, to achieve a large amount of dilution so that a is substantially smaller than g and therefore easily measurable, the weight added to the left side should be a small fraction of the initial weight.

As mentioned previously, the Atwood's machine may also be used to demonstrate the relationship between string tension and acceleration. To do this, it is first necessary to reexamine one of the equations just developed. Figure 7 repeats this relationship. The "dilution" equation above has been substituted for a in $T - m_2g = m_2a$. The resulting equation then relates tension to mass and g: the acceleration a has dropped out, of course.

Before turning to Figure 8, the student should attempt to simplify the expression given as the final step in Figure 7. When this has been done, reference may then be made to Figure 8 as a check. This equation provides the information that the tension in the string during the acceleration process may be determined from twice the product of the masses and g, divided by the sum of the masses.

$$m_1 g - \cancel{T} = m_1 a$$

$$\cancel{T} - m_2 g = m_2 a$$

ADD

$$m_1 g - m_2 g = m_1 a + m_2 a$$

$$a = \frac{m_1 - m_2}{m_1 + m_2} \cdot g$$

FIGURE ⑥

$$T - m_2 g = m_2 a$$

$$a = \frac{m_1 - m_2}{m_1 + m_2} \cdot g$$

So

$$T = m_2 \frac{m_1 - m_2}{m_1 + m_2} \cdot g + m_2 g$$

FIGURE ⑦

A relatively simple experiment may be set up to verify this statement as in Figure 9. A pair of spring balances has been inserted in the string as shown. The masses are, say, 1000 grams each. Since each balance maintains equilibrium with its particular hanging mass, first law considerations immediately dictate that each balance read 1000 x g. The g multiplier is inserted merely to keep the units correct; weight should be measured in force rather than mass units and, in this case, the weight unit should be the gram-centimeter per second per second or dyne. Alternatively, the tension may be computed from the expression given in Figure 8 by substituting 1000 grams for each mass and solving for T. This has been done in Figure 10. This calculation is equally valid for either mass, hence it shows that the tension is the same for both masses.

In the next step, a mass of 400 grams is added to the left side and the string is released while either one or both of the balances is observed during the acceleration process. It is noted that the reading in either case is 1,170 indicating that the tension is 1,170 x g dynes.

$$T = 2\ \frac{m_1\ m_2}{m_1+m_2} \cdot g$$

FIGURE ⑧

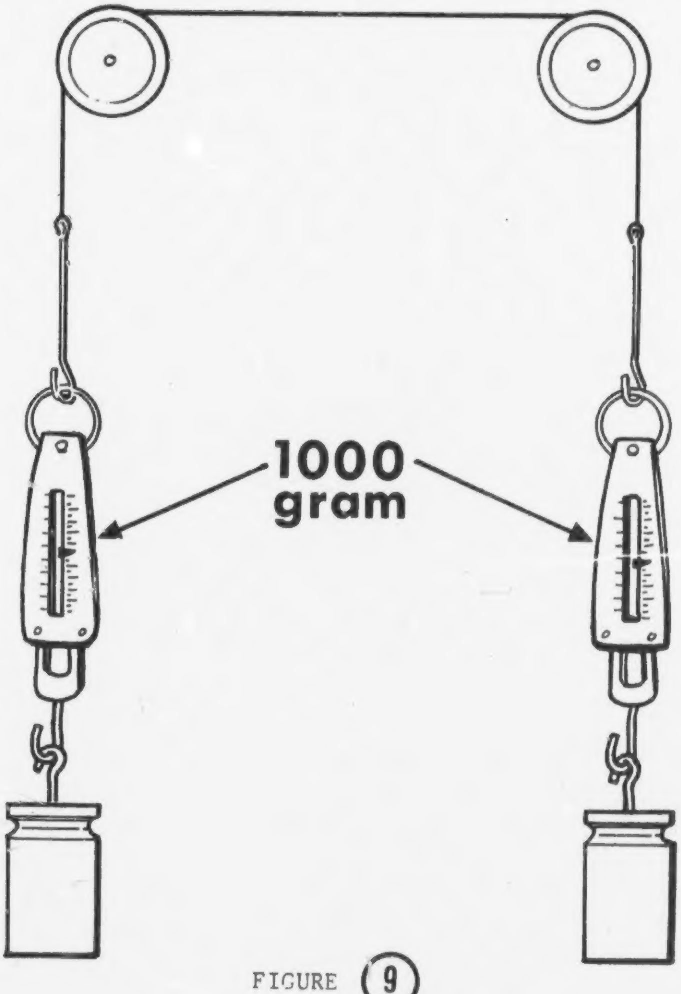


FIGURE ⑨

As a final step, the new mass value is substituted in the expression for tension shown in Figure 10. When this is done, the result agrees with the simulated experimental result just described. The student is requested to make the necessary substitutions and see for himself. Should he have difficulty in proving this out, he may refer to Figure 11 where the problem has been solved.

This discussion has attempted to present several thoughts:

(1) The Atwood's machine is capable of providing relatively precise but indirect measurements of *g*.

(2) The Atwood's machine can be used to verify the predicted tension in a string on which a mass is accelerating. In this sense, it also serves to corroborate the first and second laws of motion within the limits of error of the experimental apparatus.

$$T = 2\,\frac{m_1 \quad m_2}{m_1+m_2}\cdot g$$

$$= 2\,\frac{1000\times 1000}{1000+1000}\cdot g$$

$$= 1000\cdot g$$

FIGURE ⑩

$$T = 2\,\frac{m_1 \quad m_2}{m_1+m_2}\cdot g$$

$$= 2\,\frac{1400\times 1000}{1400+1000}\cdot g$$

$$= 1170\cdot g$$

FIGURE ⑪

ATWOOD'S MACHINE

TERMINAL OBJECTIVES

3/3 D Apply the "free body" approach to problem solutions.

Please turn to page 13A of your STUDY GUIDE to continue with your work.

CHARACTERISTICS
OF
CIRCULAR MOTION

ILLUSTRATED TEXT (1) (No Figure)

Imagine that you are a passenger in an automobile negotiating a sharp right turn. You might find yourself tending to slide along the seat toward the left. From your point of view, some force of unknown origin appears to act on your body to the left, so you invent a suitable name, calling it *centrifugal force*; that is, "centerfleeing" because it acts outward from the center of the circle you are negotiating. All things considered, you can't be blamed for doing this: you did feel this force and your body did respond to it and so it is very real to you. In actuality you were deceived by considering the motion in terms of the frame of reference of the car which is an accelerating reference frame, where Newton's laws may be so simply applied. If you look again at the situation through the eyes of an outside stationary observer, he sees that you tended to move in a straight line while the car moved along a curved path. Therefore, while the car moved to the right, it appeared to its occupant, moving with the car, that he was being thrown to the left by a force. For this reason, the centripetal force you felt as an occupant of the car is often called a fictitious force.

Please turn to Figure 1 where we consider a highly analogous situation to demonstrate the fictitious nature of the centrifugal force.

Here, imagine you are sitting on a chair which someone quickly jerks to your right. Here, too, you would feel as though you were falling to the left, although no force acts on you in that direction. Here again you might accuse a fictitious force of pushing you to the right.

To help us understand the forces involved in circular motion, let's consider other ways by which we could cause the car to take a curved path.

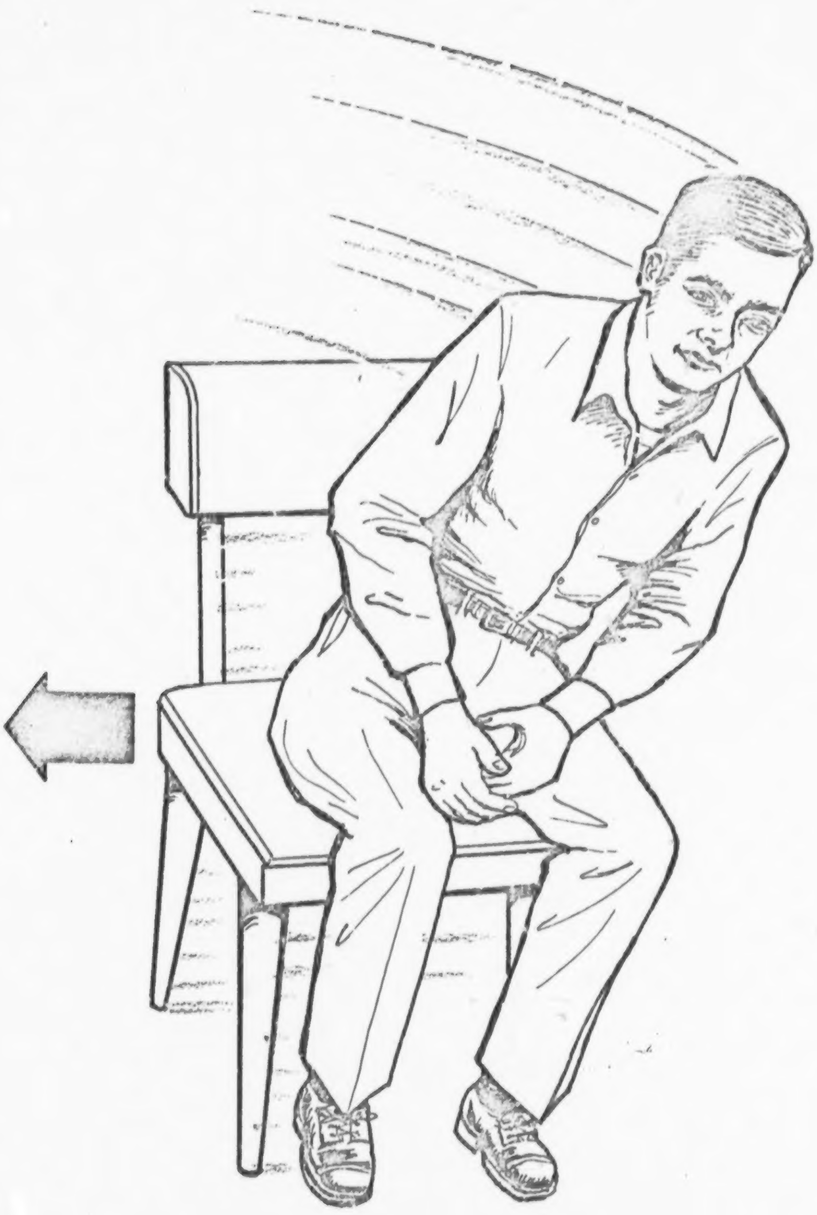


FIGURE ①

In Figure 2 we enlist the aid of a motorized toy car, with wheels fixed in the straight-on position, which we place on a tabletop. In order to curve its path we could place a fixed pole at the center of the curve and tie a string between it and the car. The string then would guide the car around the curve, by always pulling it in toward the central pole. The string then would be supplying the *inward, centripetal force,* needed to curve the path of the car, and without which the car itself would drive along a straight line at constant speed, v. When we desire to stop curving its path, we merely release the string so that the car may now proceed along its present heading - a tangent to the curve from the point where the string was released.

The same forces are acting in the case of the real car and its occupants. Inertia at any instant, wants the car and passengers to travel in a straight line at constant speed, but an inward force, the reaction to the force of the tires against the road, curves the path of the car. The passengers, however, must depend on friction against the seat to pull them into the same curved path.

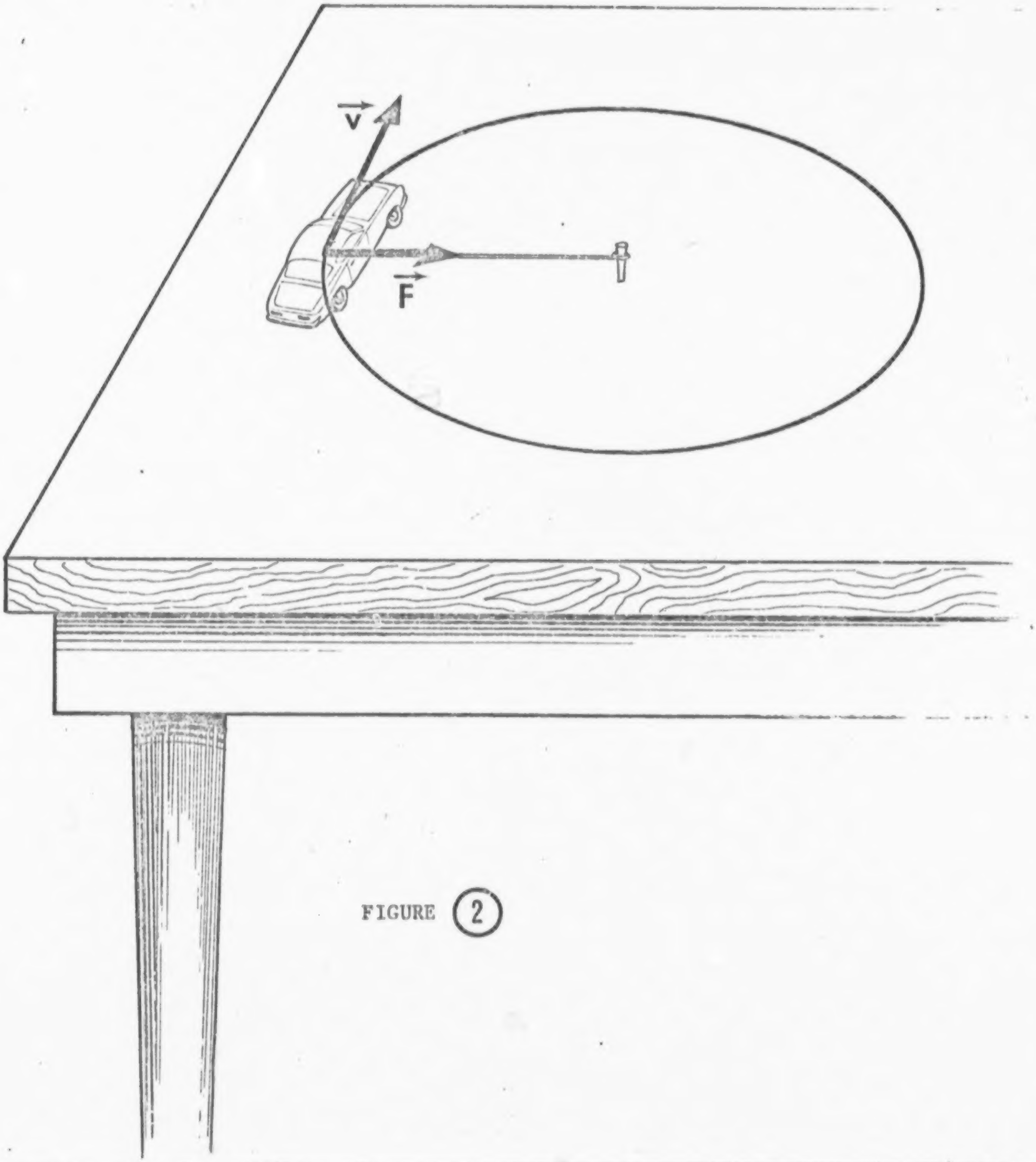


FIGURE ②

While the apparent centrifugal force discussed earlier is fictitious and is due to the accelerating reference frame, there really are centrifugal forces occurring in this problem. They are the reactions to the centripetal forces we find. For instance, in the case of the string guiding the car, it pulls the car *inward* (centripetally) and at the same time pulls the post *outward* (centrifugally). While the post is fixed, and therefore does not undergo an an acceleration, the car is free to respond to this force and its path is, therefore, curved.

Countless other examples of circular motion may be observed. In Figure 4 you can see one which is becoming more and more common.

CENTRIPETAL & CENTRIFUGAL FORCES IN CIRCULAR MOTION

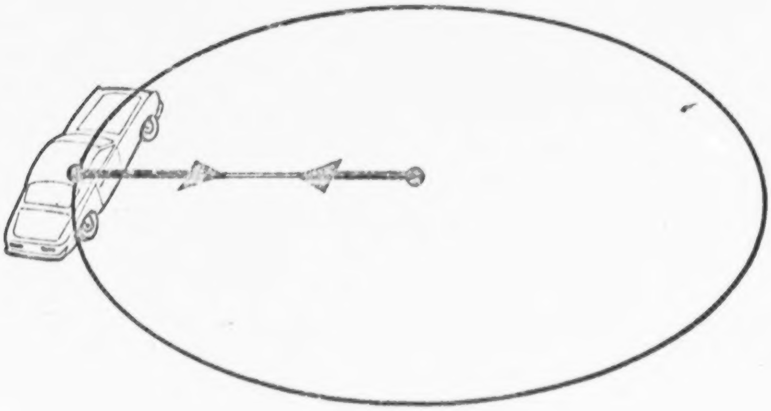


FIGURE ③

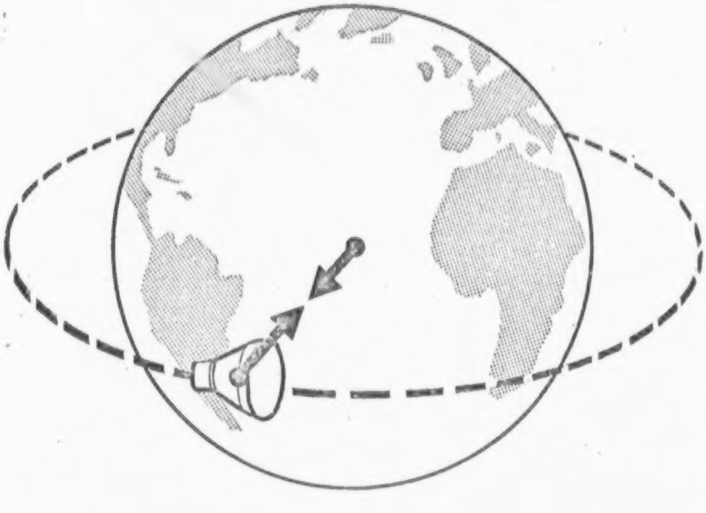


FIGURE ④

As long as a centripetal force acts, the path of the car and thus its velocity changes: that is, the car accelerates centripetally in the direction of the force, given by the equation

$$a_c = \frac{v^2}{r}$$ as shown in Figure 5

Substitution of centripetal acceleration into Newton's Equation of Motion, $F = ma$, we find

$$F_c = \frac{mv^2}{r}$$

Centripetal Acceleration

$$a_c = \frac{v^2}{r}$$

Substituted into the Equation of Motion

$$F = ma$$

Yields an Equation for Circular Motion

$$F_c = \frac{mv^2}{r}$$

FIGURE ⑤

Work When Force Varies In Both Magnitude & Direction

Fundamentally, work is a product of a force and a displacement. If the force is constant throughout the displacement, the problem of determining the work done by the force is a simple one. However, since force is a vector quantity it may vary in magnitude, direction, or both and, should this variation occur during the time of the displacement, the task of finding the work done naturally becomes more complex. An understanding of the procedure to be used in calculating work is best attained by moving through a series of examples starting with the simplest type and gradually introducing the possible variations.

(Figure 1) An inclined plane making an angle of 30° with the horizontal carries a block on which a force F acts. The plane is to be considered frictionless, hence, the force F produces an acceleration of the block up the incline. As a result of the action of F, the block is displaced from position s_1 to position s_2. The problem is to find the work done by the force F over the distance from s_1 to s_2.

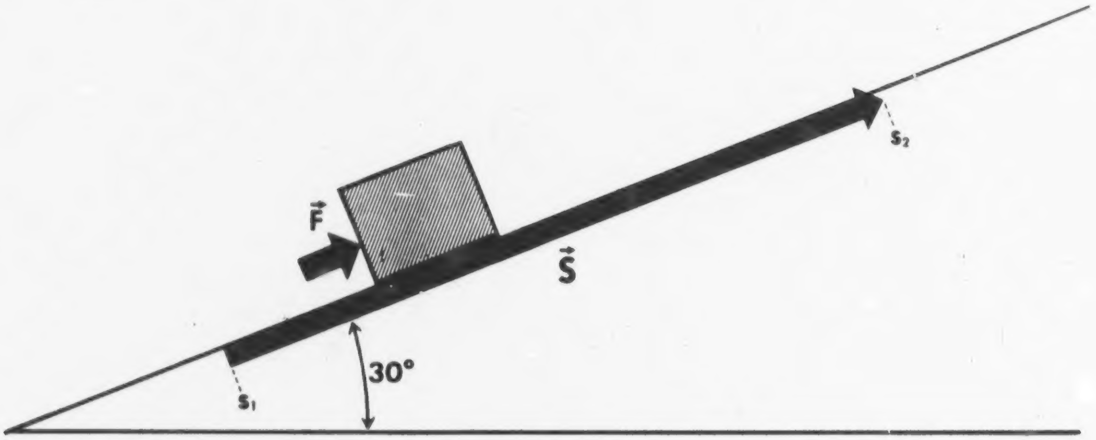

$\vec{F}$
$\vec{S}$
s_2
s_1
30°

FIGURE ①

(Figure 2) One way to solve the problem is to plot the force against the displacement as shown in Figure 2. The force appears on the y-axis which is labeled F_s to indicate that the force acts along the path over which the displacement occurs -- that is, parallel to the incline of the plane. Assuming the force to be constant throughout the displacement, it is plotted as a straight horizontal line parallel to the x-axis from s_1 to s_2. Since the total displacement is $(s_2 - s_1)$, from the basic definition of work it is seen that the work done is $F(s_2 - s_1)$ -- a scalar product, the force F has therefore accomplished a definite amount of work in moving through the distance $s_2 - s_1$.

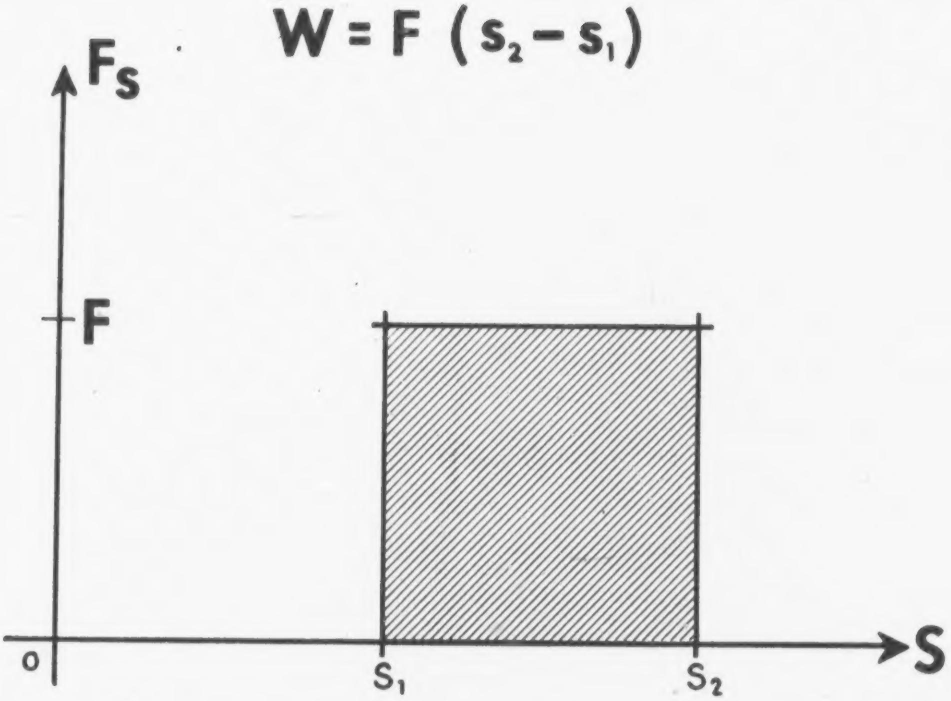


FIGURE ②

(Figure 3) Note the new condition introduced in this drawing. The force F is no longer parallel to the plane: instead, it is horizontal, making an angle of 30° to the line of the incline. The force exerted in this direction would again cause the block to accelerate but, as might be anticipated, the acceleration would not be as great as it was in the previous example for a force of the same magnitude. In this case, only the component of F parallel to the plane contributes to the acceleration and, of course, this component is smaller than F itself so that one would not expect the acceleration to be as great. To determine the work done by F under these changed conditions, it is necessary to calculate the magnitude of the component of F parallel to the plane, that is F_s, since only this component is involved in the work process.

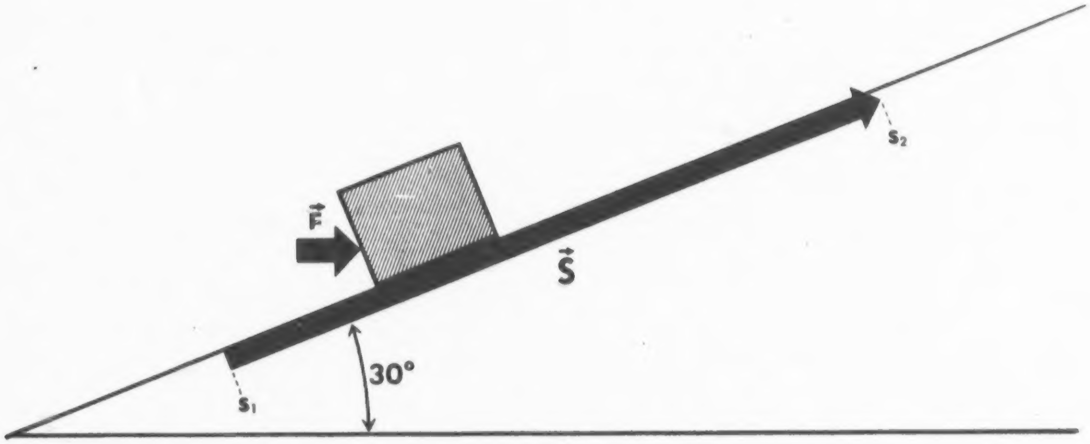

$\vec{F}$
$\vec{S}$
s_2
s_1
30°

FIGURE ③

(Figure 4) This is a plot of F_s versus s once again but here F_s is the component of F parallel to the plane, that is, $F_s = F \cos \theta$. The component perpendicular to the plane does not contribute to the work, hence it is omitted from consideration altogether. The angle θ between the applied force and its useful component is the same as the angle of incline as is easily proved by elementary geometry. With the magnitude of F constant and with the angle remaining unchanged throughout the displacement, then $F \cos \theta$ is also a constant. Once again the scalar value of the work is merely the product of the useful component of the force and the displacement or

$$W = F \cos \theta \, (s_2 - s_1) \quad \text{(The area under the curve)}$$

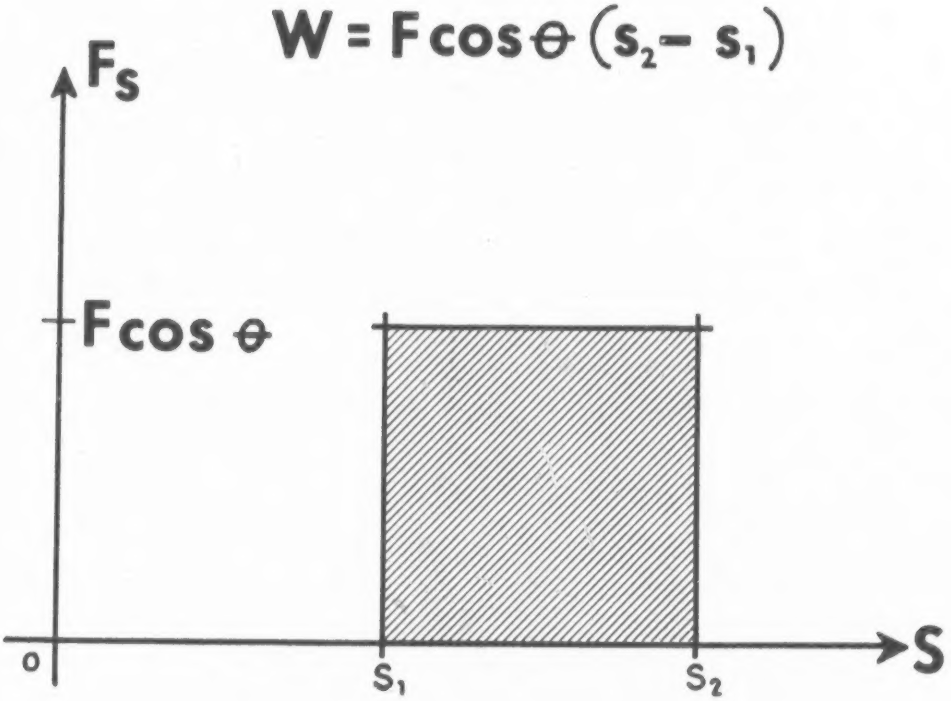


FIGURE ④

(Figure 5) The analytical approach is given in Figure 5. On the first line is the general expression for the work W as the integral from s_1 to s_2 of $\vec{F} \cdot \vec{ds}$. When this dot product is expanded, the expression given in the second line is obtained in which F cos $\theta_{F,s}$ is the component of the original force F in the s-direction. The term ds, of course, is the incremental displacement along which F is acting.

Applying this to the simplest case as discussed above, F is constant and when it is parallel to the plane, the angle θ is zero, hence the cosine of the angle is unity. The constant F may be moved to the left of the integral sign, cos θ dropped, to yield the expression given in line 3. The integral of ds is simply s, so that the evaluation proceeds as in lines 4 and 5. Note that this is exactly the same expression as was formerly obtained by using the area under the F-s curve.

$$W = \int_{S_1}^{S_2} \vec{F} \cdot d\vec{s}$$

$$= \int_{S_1}^{S_2} F \cos\theta_{F,S}\, ds$$

$$= F \int_{S_1}^{S_2} ds$$

$$= F \quad S \Big|_{S_1}^{S_2}$$

$$= F \left(S_2 - S_1 \right)$$

FIGURE ⑤

(Figure 6) This development is based on the second example in which F is horizontal rather than parallel to the plane. The first two lines are self-explanatory. In the third line, F cos θ has been moved to the left of the integral sign since both are constant, and the subscript "I" has been added to the θ to indicate that this is the angle of the inclined plane. Evaluating the integral as in lines 4 and 5, it is seen that the final expression for the work done is identical with that which emerged from the geometric analysis above.

$$W = \int_{S_1}^{S_2} \vec{F} \cdot \vec{ds}$$

$$= \int_{S_1}^{S_2} F \cos\theta_{F,S}\, ds$$

$$= F \cos\theta_I \int_{S_1}^{S_2} ds$$

$$= F \cos\theta_I \, S \Big|_{S_1}^{S_2}$$

$$= F \cos\theta_I (S_2 - S_1)$$

FIGURE ⑥

(Figure 7) This is the same diagram as in Figure 1, but a new element is to be supplied by the reader's imagination. Let the force F increase in magnitude at a steady rate as the block is moved from s_1 to s_2. With the force increasing in this way, its magnitude is clearly some function of the displacement; as a matter of fact, the function must be a linear one if the increase occurs at a uniform rate as stipulated. This means that the relationship between F at any instant of the displacement must be related to the displacement by a proportionality constant k. That is, $F = ks$.

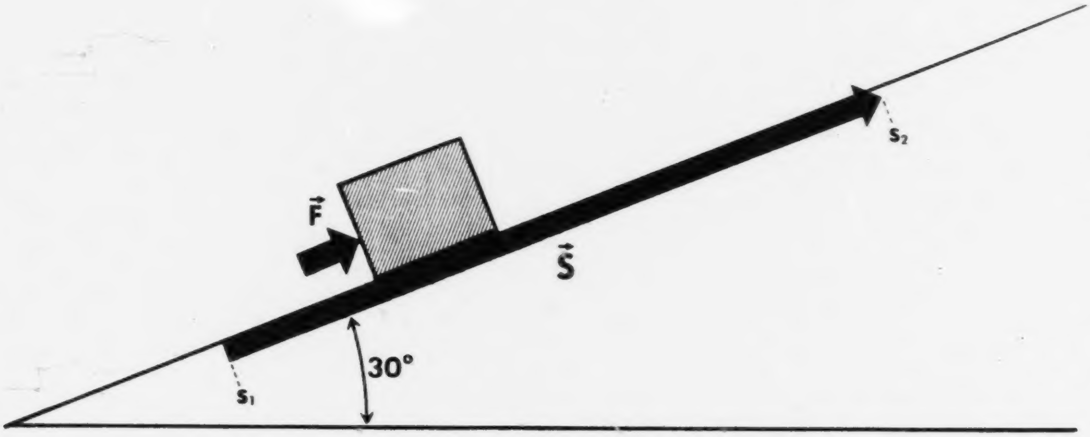

$\vec{F}$
$\vec{S}$
s_2
s_1
30°

FIGURE ⑦

(Figure 8) When such a varying force is plotted against displacement, the graph shown in Figure 8 results. The angle made by the applied force is still constant but it may have any value at all since $F_s = F \cos \theta$, but the term F_s has been replaced by ks as previously explained since the force is now a function of displacement. The graph must be a straight line starting at the origin because F must be zero when s is zero, and its positive slope indicates that the force increases with displacement.

The student is now earnestly requested to set up the required integral for determining work using the procedural pattern shown in Figures 5 and 6. He is to solve the integral for a general expression giving work in terms k, θ, and s without looking ahead in the text. Only after this has been attempted should he proceed.

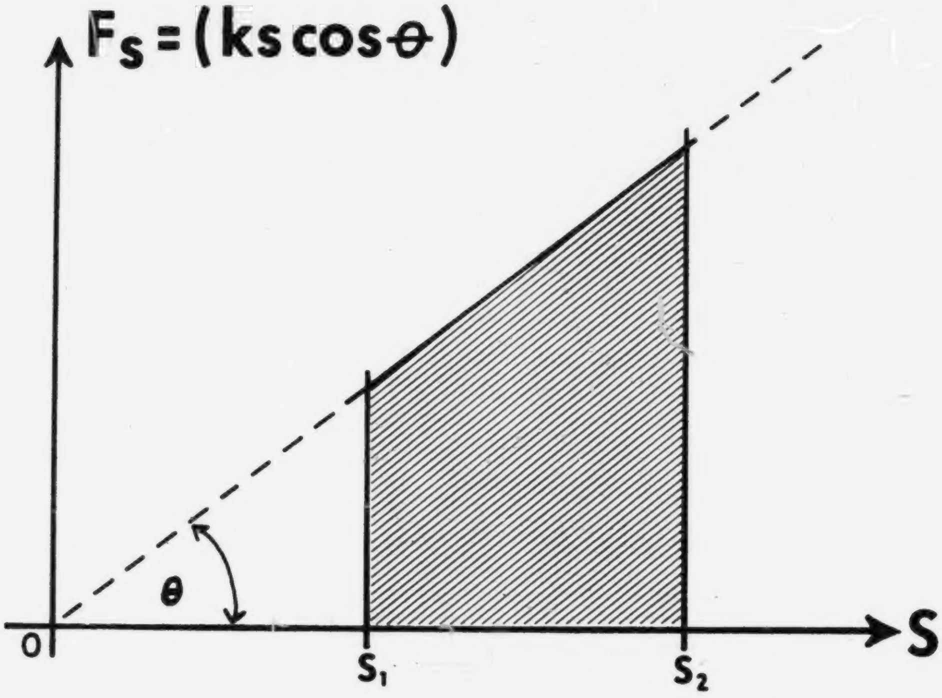


FIGURE ⑧

(Figure 9) The solution to the problem appears in Figure 9. It should be checked against the student's approach to verify the accuracy of the work or to locate the correct errors. Another valuable step at this point in the work is to work out the equation by the geometric method involving the determination of the area under the curve bounded by s_1 and s_2 in Figure 8. To find the area of the trapezoid, find the area of the base rectangle and add to this the area of the remaining triangle. When properly handled, this method will yield the same expression for work done, or

$$W = \frac{1}{2} k \cos \theta \left(s_2^2 - s_1^2\right)$$

The final item in this discussion deals with the calculation of work when the force varies in both magnitude and direction.

$$W = \int_{s_1}^{s_2} ks \cos \theta \, ds$$

$$= k \cos \theta \int_{s_1}^{s_2} s \, ds$$

$$= \frac{1}{2} k \cos \theta \, (s_2^2 - s_1^2)$$

FIGURE ⑨

(Figure 10) The student must now imagine that F is not only growing in magnitude but is also changing in direction in some steady manner as the block moves up the plane. To calculate the work for a complex action like this, it is necessary to know how the components of F vary with position, or to have an expression that gives the relationship between the component of F parallel to the plane and the displacement itself. One such possible relationship would have it that F_s, the parallel component, is directly proportional to the square of the displacement or

$$F_s = ps^2 \qquad \text{where } p = \text{constant}$$

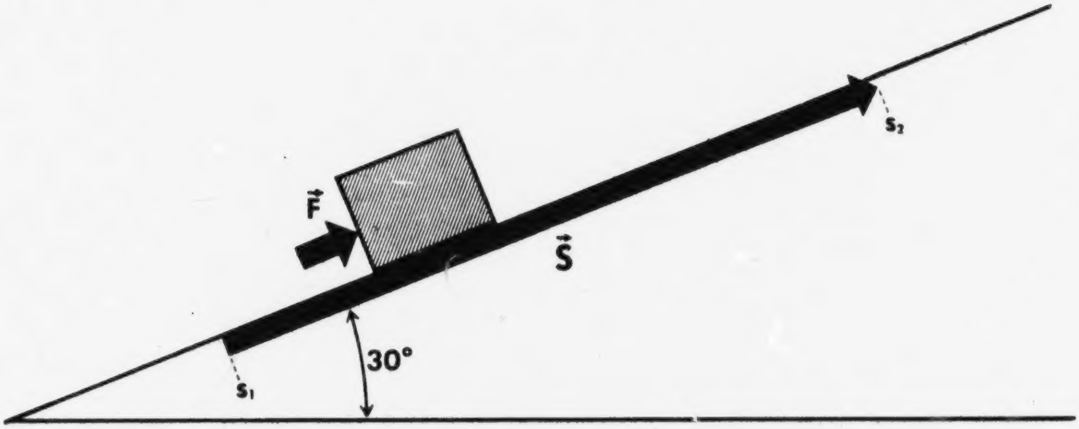

$\vec{F}$
$\vec{S}$
s_2
s_1
30°

FIGURE ⑩

(Figure 11) In this figure, F_s $(=ps^2)$ has been plotted against displacement. The resulting curve is a parabola as might have been expected from the equation. Using a procedure identical with that of the previous examples, the work may be calculated by setting up and solving the proper integral equation.

The student is again asked to set up this equation and evaluate it in general terms before proceeding to the conclusion of this text. He may then go on to the next figure.

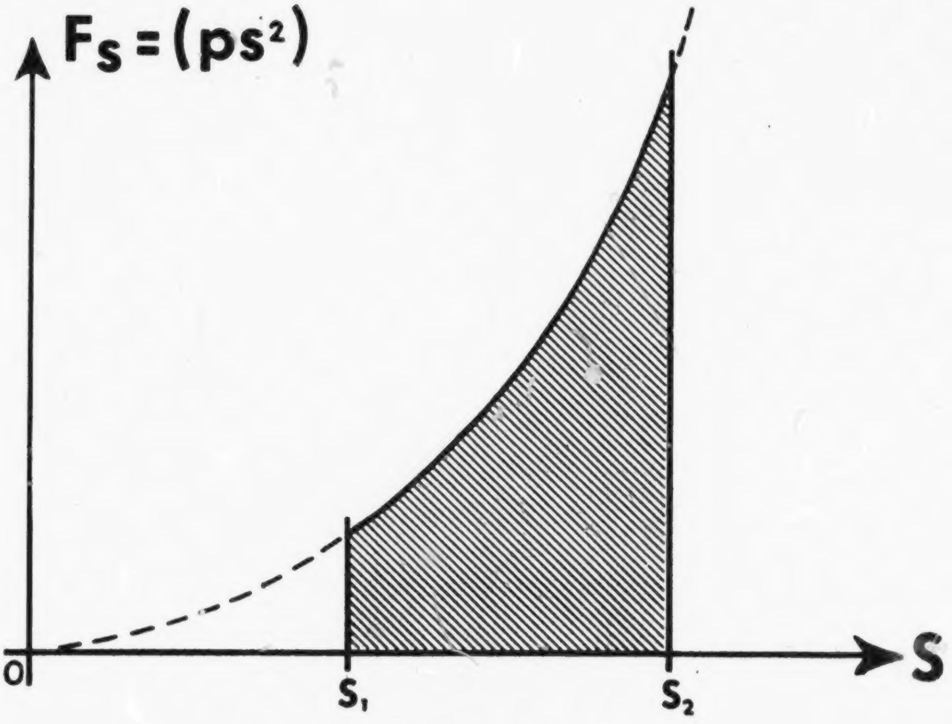


FIGURE ⑪

(Figure 12) This is the general equation required for finding the work when the force varies in both magnitude and direction according to the relationship $F_s = ps^2$. The final evaluation expression that should be obtained is

$$W = \frac{1}{3} ps_2{}^3 - \frac{1}{3} ps_1{}^3$$

$$W = \int_{S_1}^{S_2} ps^2 \, ds$$

FIGURE ⑫

Work When Force Varies In Both Magnitude & Direction

TERMINAL OBJECTIVES

5/1 B Calculate work associated with variable forces.

Please turn to page 26A of your STUDY GUIDE to continue with your work.

KINETIC ENERGY

The damage inflicted on one or the other of two objects that collide at high relative speed depends to a very great extent on the magnitude of the relative velocity. Since the physical quantity known as kinetic energy is, in turn, a function of velocity, an interrelationship between impact damage and kinetic energy exists. It is the purpose of this discussion to derive a quantitative statement which provides information relative to this relationship.

Approaching the problem from first principles, Newton's second law of motion may be expressed quantitatively in the form shown in Figure 1. In the vector equation given first, the acceleration term $\underline{\vec{a}}$ may be replaced by the rate of change of velocity $d\vec{v}/dt$ so that the form of the equation obtained becomes $\vec{F} = m\, d\vec{v}/dt$. This form of the second law will be used in this discussion.

The work done on a body is given by $\int \vec{F} \cdot d\vec{s}$ (Figure 2), in which $\vec{F}$ is the resultant force acting on a body and $d\vec{s}$ is an element of distance over which the body moves as a result of the unbalanced force acting on it.

$$\vec{F} = m\vec{a}$$

$$= m\frac{d\vec{v}}{dt}$$

FIGURE ①

WORK DONE ON BODY

$$= \int \vec{F} \cdot d\vec{s}$$

$\vec{F} =$ resultant force on body

FIGURE ②

Figure 3 illustrates how the second law and work statement may be combined in a single expression. The force term has been replaced with m $d\vec{v}/dt$ and the limits of integration (from s_1 to s_2) have been inserted. The "dt" term in the expression shown may be considered as a simple divisor in the fraction and may, therefore, be shifted to a different position as shown in Figure 4.

The advantage gained by shifting this term is apparent: since $d\vec{s}/dt$ represents the velocity of the body, it is now possible to rewrite the equation in the form illustrated in Figure 5. Here, the work done on the body is expressed in terms of mass and velocity, the displacement having been eliminated. To accommodate the new form, the limits of integration may now be changed from displacement to velocity as indicated in Figure 6.

This integral is quite easy to evaluate. The reader should perform this integration for himself before turning to the solution given in Figure 7. The integral of $m\vec{v}\cdot d\vec{v}$ is $mv^2/2$. With the substitution of the limits, the expression finally becomes

$$\text{Work done} = W = \frac{1}{2}\,mv_2^2 - \frac{1}{2}\,mv_1^2$$

The quantity $\frac{1}{2}\,mv^2$ is an entity of substantial importance in physics; it is called kinetic energy. Thus, the statement above may be verbalized by saying that the difference between initial and final kinetic energies of a body is equal to the work done on the body to bring about this change of kinetic energy. This is summarized in Figure 8. It is an important result and well worth noting.

$$W = \int_{S_1}^{S_2} \vec{F} \cdot d\vec{s}$$

$$= \int_{S_1}^{S_2} m \, \frac{d\vec{v}}{dt} \cdot d\vec{s}$$

FIGURE ③

$$W = \int_{S_1}^{S_2} \vec{F} \cdot d\vec{s}$$

$$= \int_{S_1}^{S_2} m \, d\vec{v} \cdot \frac{d\vec{s}}{dt}$$

FIGURE ④

$$W = \int_{S_1}^{S_2} \vec{F} \cdot d\vec{s}$$

$$= \int_{S_1}^{S_2} m \, d\vec{v} \cdot \vec{v}$$

FIGURE ⑤

$$W = \int_{S_1}^{S_2} \vec{F} \cdot d\vec{s}$$

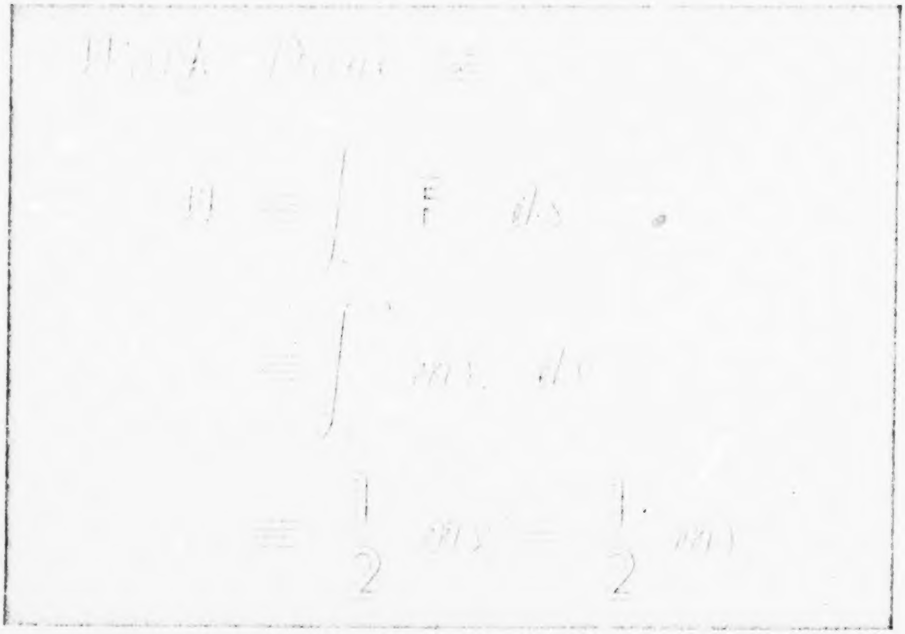
$$= \int_{S_1}^{S_2} m \, d\vec{v} \cdot \vec{v}$$

$$= \int_{V_1}^{V_2} m \, \vec{v} \cdot d\vec{v}$$

FIGURE ⑥

Under the proper conditions, this process is fully reversible. That is, if an object already possesses kinetic energy it then has the capability of doing work; for example, when the kinetic energy of a fast-moving automobile is expended in a collision with a sturdy tree, work is done on both the tree and the automobile. This work generally takes the form of a gashed tree-trunk and a demolished car!

The expression in Figure 8 contains another implication that is extremely important both in physics and in our daily lives. The kinetic energy of any moving body is a function of the square of the velocity. An automobile moving at a speed of 20 mi/hr has an easily calculated kinetic energy and, consequently, the capability of doing a given amount of damage if it is brought to rest in a collision. When the speed is increased to 40 mi/hr, however, the kinetic energy -- hence the capability for inflicting damage --- quadruples. At 60 mi/hr this capability is 9 times as great as at 20 mi/hr, and at 80 mi/hr it is 16 times as great!



Work Done =

$$W = \int F \, ds$$

$$= \int m v \, dv$$

$$= \frac{1}{2} mv - \frac{1}{2} mv$$

FIGURE ⑦

Work Done = Change in Kinetic Energy

$$\int F \, ds = \frac{1}{2} mv - \frac{1}{2} mv$$

FIGURE ⑧

KINETIC ENERGY

TERMINAL OBJECTIVES

5/2 D Answer qualitative questions about Kinetic Energy.

Please turn to page 38A of your STUDY GUIDE to continue with your work.

POTENTIAL ENERGY

Modern buildings of glass and stainless steel embrace every modern scientific technique to provide pure air, good lighting, many other material comforts. But despite the emphasis on the new and the modern, relatively primitive methods are still being used to tear down the original structures. Perhaps these methods persist because they work; perhaps they are economical and fast. In any case, it is not uncommon to see an ancient wrecking-ball crane in action along the streets of New York City. (Figure 1)

The wrecking-ball is not unlike the battering ram used by the Romans. It hangs on a steel cable attached to a horizontal crane arm. The crane operator gets the ball swinging by oscillating the arm and, when the swing is wide enough, he brings the arm quickly toward the building causing the ball to crash into the wall. Since the ball is very massive, it develops an enormous amount of kinetic energy at the instant of impact. If the ball loses most of its speed on impact, its kinetic energy is largely converted into physical work.

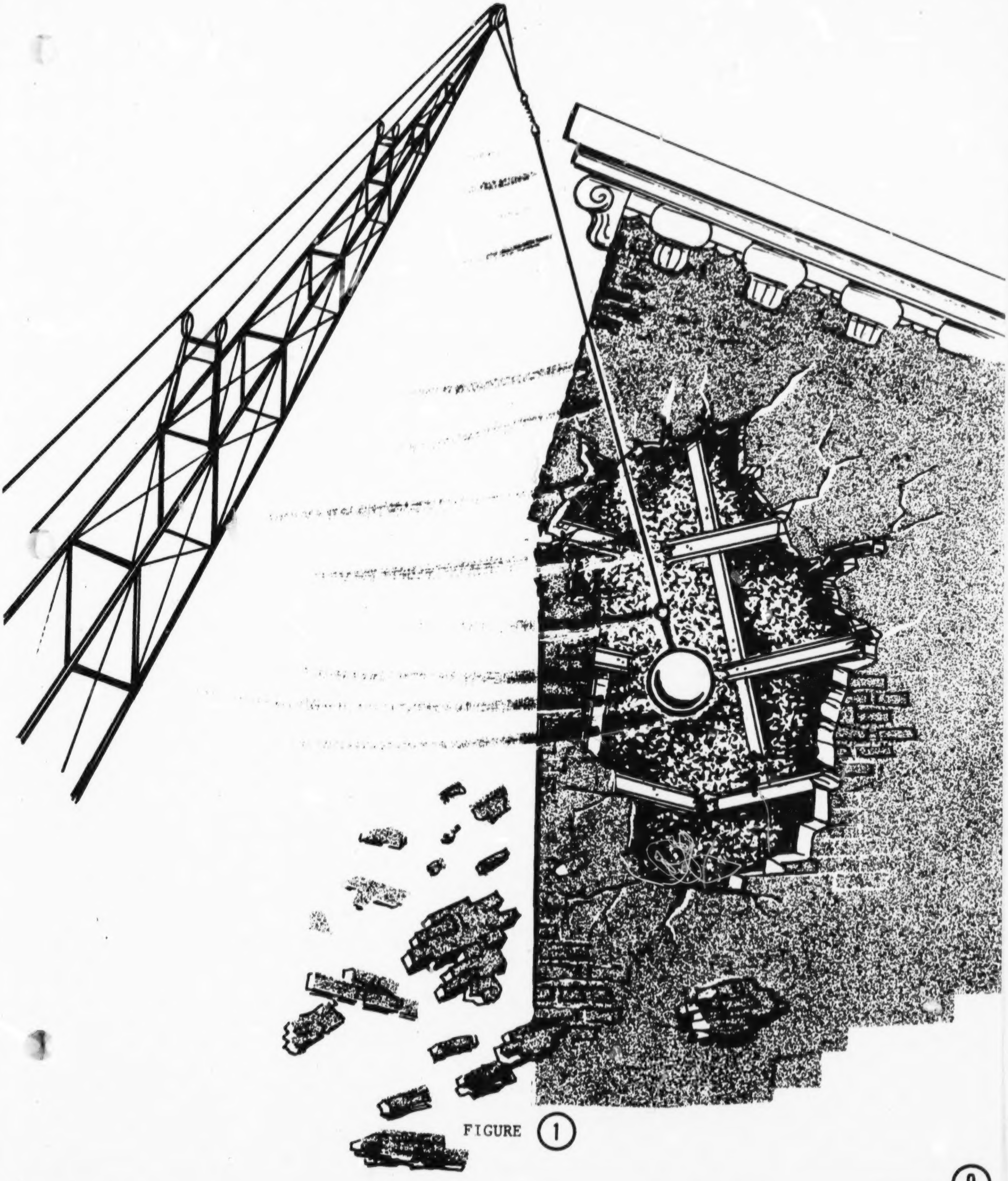


FIGURE ①

A pile driver is another of the primitive devices mentioned above; it is used to drive wood pilings into the ground to provide added support for a building foundation. (Figure 2)

It consists of a massive head or hammer that is raised to the top of a supporting structure. When the head is released and allowed to fall, it strikes the top of the pile and comes to rest, exerting tremendous force and doing a substantial amount of work.

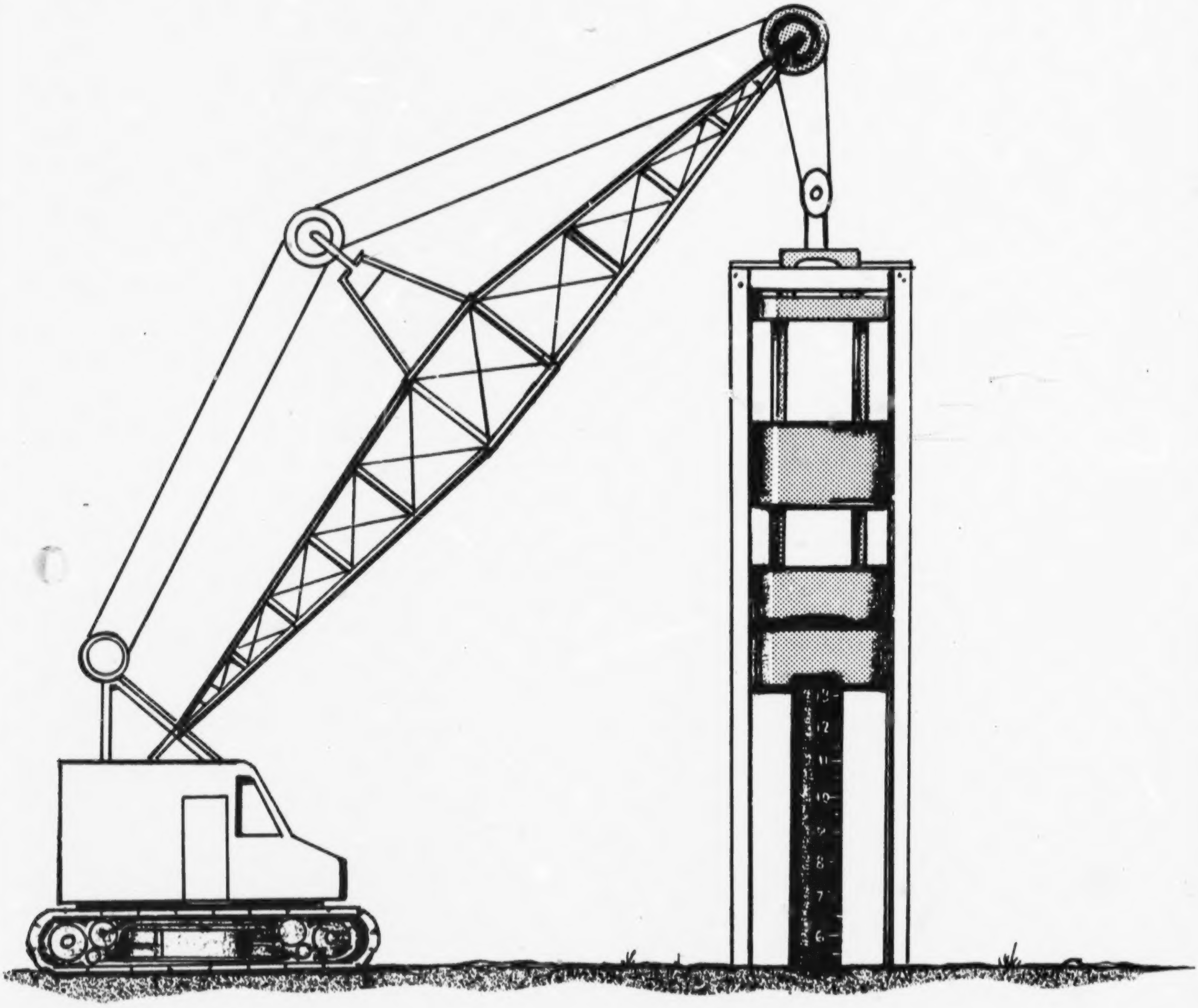

13
12
11
10
9
8
7
6

FIGURE ②

The difference between these two sequences should be emphasized. In the case of the wrecking ball, the operator imparts kinetic energy to the ball through the medium of his engine; the operator of the pile driver, however, merely causes the ball to be lifted from a low position to a higher one. Note that his engine does not directly impart kinetic energy to the hammer. On the other hand, since the hammer was capable of doing work as a result of the efforts of the pile driver engine, it is reasonable to conclude that these efforts did result in some kind of energy storage in the hammer. While in the stationary raised position, the hammer has no velocity, hence no kinetic energy. Yet energy has been stored in it by virtue of its raised position otherwise it could not have done work at a later time. Since the hammer has the capacity to do work, it has energy. (Figure 3)

Potential energy is the energy of position or state. For the pile driver, position is the important aspect of the change that occurred. The work done by the pile driver engine is converted to the potential energy of position when the hammer is raised to the top of the tower structure.

POTENTIAL ENERGY

= *Energy of*
Position
or state

FIGURE ③

Potential energy may be stored in other ways. When a spring is compressed or stretched the work done in the process is converted mainly into the potential energy of changed state. An arrow drawn back on the bowstring changes the state of the bow so that, in bending, it possesses potential energy it did not have initially. Explosives have potential energy of state, too; in this case, the change of state is chemical in nature. Some time in the past an energetic agency like the sun brought about changes which have stored explosive power in the resulting compounds. The quantitative aspects of potential energy may be approached through an example using a spring. (Figure 4)

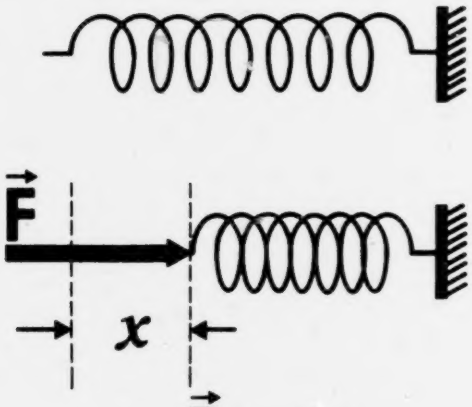
The spring shown in the upper drawing is unloaded, neither compressed nor stretched. If the spring is compressed so that its end moves over a distance x, the force required for the compression may be given as $k\vec{x}$ in which k is the spring constant (Hooke's Law). In the diagram, $\vec{F}$ is directed toward the right. As the compression proceeds, an increasing force is required to overcome the resistance offered by the elasticity of the spring. The magnitude of the force needed to produce a specific displacement $d\vec{x}$ is therefore a function of x itself, hence $\vec{F}$ is variable. Refer to Figure 5.

If the displacement of the end of the spring is to be from x_1 to x_2, then the work required to cause this displacement is given by the integral of F dx between the limits x_1 and x_2. The vector notation may be dropped at this point because the force and the ensuing displacement are in the same direction. This is shown in Figure 6.

It has already been shown that the applied force may be given as kx, hence kx may be substituted for F in the scalar equation as indicated in Figure 7.

The integration may now be performed. The integral of kx dx is $1/2\ kx^2$. Substitution of the limits yields the final expression shown in Figure 8.

Thus, the work done in compressing a spring from one x position to another is the difference between the $1/2\ kx^2$ values for the two positions. The reader should bear in mind that this equation specifically applies to the distortion of a spring, a case where the force required is variable.



$$\vec{F} = k\vec{x}$$

FIGURE ④

$$W = \int_{x_1}^{x_2} \vec{F}\, d\vec{x}$$

FIGURE ⑤

$$W = \int_{x_1}^{x_2} F\, dx$$

FIGURE ⑥

$$W = \int_{x_1}^{x_2} F\, dx$$
$$= \int_{x_1}^{x_2} kx\, dx$$

FIGURE ⑦

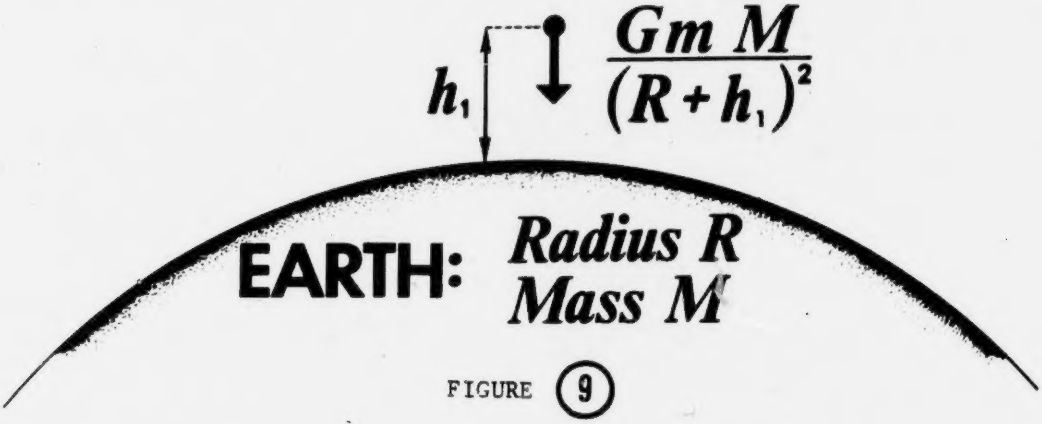
$$W = \int_{x_1}^{x_2} F\, dx$$
$$= \int_{x_1}^{x_2} kx\, dx$$
$$= \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

FIGURE ⑧

Refer to Figure 9. Here is a different kind of situation in which potential energy is also involved. When a mass is raised to a higher position from an initial low position over a relatively short distance, say, one hundred feet or so, the force of gravity that resists this action changes so little that the change may be ignored without introducing significant error. For this limited case the force needed to raise the mass against gravity may be considered to remain constant throughout the action. In this diagram the mass is shown to have a weight mg close to the surface of the earth.

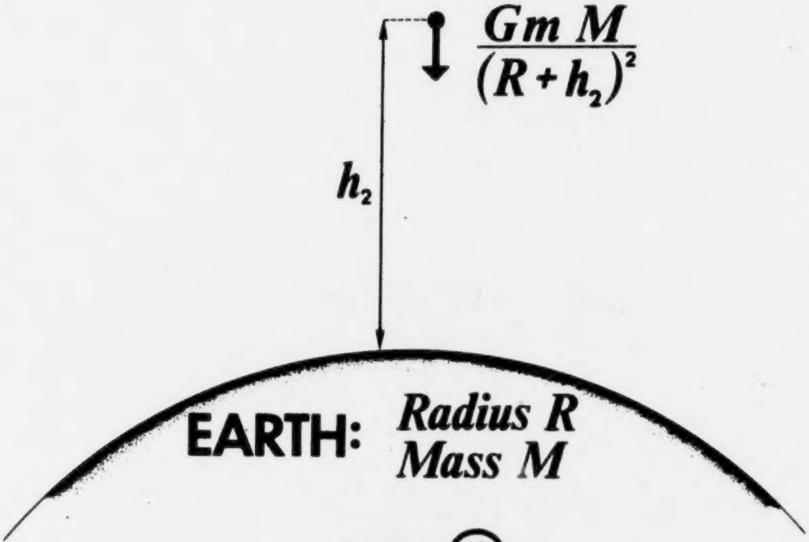
In Figure 10 the weight is shown raised over a distance h to some higher position. The force required to raise the mass is equal in magnitude to mg, the weight of the body. The work done in this case is merely the product of the force and the displacement since the force is constant.

(Figure 11) The work done in raising the mass to the new position must result in a change of potential energy equal to mgh. This is a special case of a change in gravitational potential energy in contrast with the previous example where compression resulted in a change in elastic potential energy. These ideas may be summarized as follows: the change in potential energy of a body is equal to the work done in moving the body (initially at rest) from one position or state to a second position or state where it is also at rest. You have seen that the force of gravity can be considered to be constant when a mass is raised a short distance above the earth. But if the distance through which the body is raised is large, the gravitational force gradually decreases. For this condition, the gravitational force must be considered as variable rather than constant.



h_1
$\frac{Gm\,M}{(R+h_1)^2}$
EARTH:
Radius R
Mass M

FIGURE ⑨



$\frac{Gm\,M}{(R+h_2)^2}$
h_2
EARTH:
Radius R
Mass M

FIGURE ⑩

CHANGE IN POTENTIAL ENERGY

$$= \int \vec{\mathbf{F}} \cdot d\vec{s}$$

$$= \int_{h_1}^{h_2} \frac{Gm\,M\,dh}{(R+h)^2}$$

FIGURE ⑪

For example, suppose a mass m is located at a height h_1 above the earth's surface. It's weight from the Law of Universal Gravitation would then be a function of G, the constant of gravitation, m, and M -- the mass of the earth -- as well as the radius of the earth R and the height h_1. (Figure 12)

Next, imagine that it is raised to a new height h_2 which is considerably further from the surface than h_1. In this case, the weight would be smaller than before and would be given by the relation shown in Figure 13. The work needed to accomplish this must now be found by integrating F ds between the limits of h_1 and h_2. (Figure 14)

The correct transition is shown in this figure. The force F is replaced by its equivalent $GmM/(R + h)^2$, and ds is replaced by dh. It is left as an exercise for the student to carry the integration out to its conclusion and arrive at a general equation for the work required to raise a mass from one level to another with respect to the earth when the distance involved is sufficiently great.

Briefly, then, the change in potential energy of a body is equal to the work done in moving it from some initial rest position to some other final rest position. The change in potential energy can be determined from the product of the force and the displacement in the direction of the force if the force is constant. Finally, the change in potential energy when the force varies must be found by integrating all the elemental changes in F over the distance through which it must move in producing the displacement.

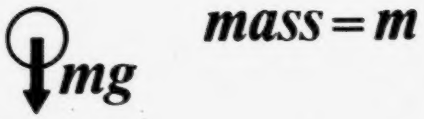


FIGURE ⑫

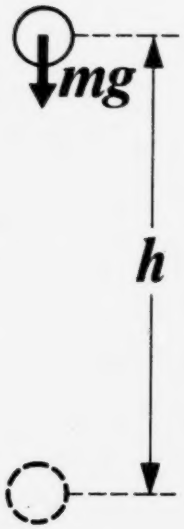


FIGURE ⑬

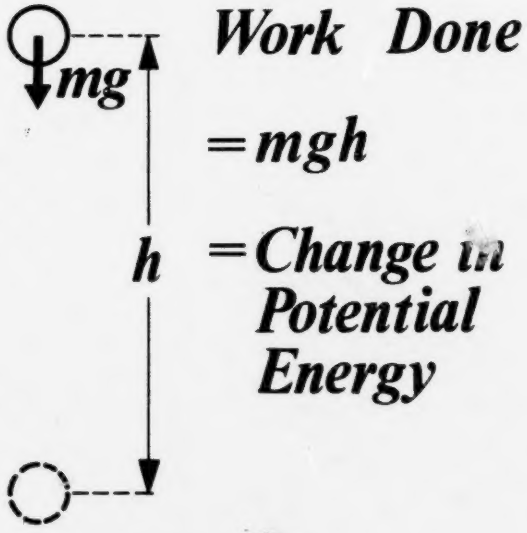

mg
h
Work Done
= mgh
= Change in Potential Energy

FIGURE ⑭

POTENTIAL ENERGY

TERMINAL OBJECTIVES

5/3 A Use the concept of potential energy for objects near the surface of the Earth and for springs.

Please turn to page 33A of your STUDY GUIDE to continue with your work.

CONSERVATION
OF
ENERGY

Physics has many aspects and many paths to follow if one is to realize a well-rounded training in the subject. Yet despite its ramifications, there are certain unifying principles that one consistently encounters along almost all of the paths. One of these certainly is the Principle of Conservation of Energy. This principle is so fundamental and so far-reaching that the student must make every effort to understand all of its implications and applications, as well as the statement of the principle itself. The ability to say, "Energy can neither be created nor destroyed but only changed in form" does not signify comprehension nor the ability to solve practical problems in which conservation is involved. Only by careful analysis followed by conscientious practice can this ability be developed.

It would be well to begin the analytical treatment with a review of some concepts that have been previously introduced. Suppose that an external, unbalanced force is applied to a free body. The work done on the free body will be equal in magnitude to the change of momentum of the body multiplied by the displacement due to the action of the force. Figure 1 shows why this statement is justified. The total work done on the body is the integral of F dx. But from the second law of motion, it is known that any force may be replaced by the change of momentum it produces. Thus, it is perfectly valid to say that the integral of F dx may be replaced by the integral of the change of momentum multiplied by the displacement dx. Thus,

$$\text{Work done} = \int F\ dx = \int m \frac{dv}{dt}\ dx$$

However, when the right-hand expression is integrated it becomes $\Delta \frac{1}{2} m v^2$, or the change in kinetic energy of the body to which the force has been applied. The relationship is given verbally in Figure 2.

$$\int F \, dx$$

$$= \int m \frac{dv}{dt} \, dx$$

$$= \Delta \frac{1}{2} m v^2$$

FIGURE ①

WORK DONE
= CHANGE IN
KINETIC ENERGY
(free body)

FIGURE ②

Next consider a situation in which a force is applied to an object which then moves under conditions such that a restoring force appears as a result of the motion. When a force is applied to a mass in a gravitational field to lift it against the pull of gravity, the object will move to a new position and stay there only as long as the original lifting force is present. When the lifting force is removed, the restoring force brings the object back to its initial position.

A similar situation exists when the spring of a balance, as in Figure 3, is stretched by an external force. As soon as the hook of the balance begins to move to the right, the spring begins to exert a restoring force that tends to bring the hook back to its starting position when the external force disappears. A completely analogous action occurs when a spring is compressed by an applied force. The work done in compressing a spring is again given by the integral of F.dx. Here, F may be replaced by k_2x, where k is the spring constant. Integrating this expression yields 1/2 kx, the potential energy of the spring after it has been compressed over a distance x from an initially uncompressed state. This is summarized in Figures 4 and 5. So it is seen that under certain conditions, the work done on a body may become the change in its kinetic energy and that under other conditions, the work done may be converted into potential energy.

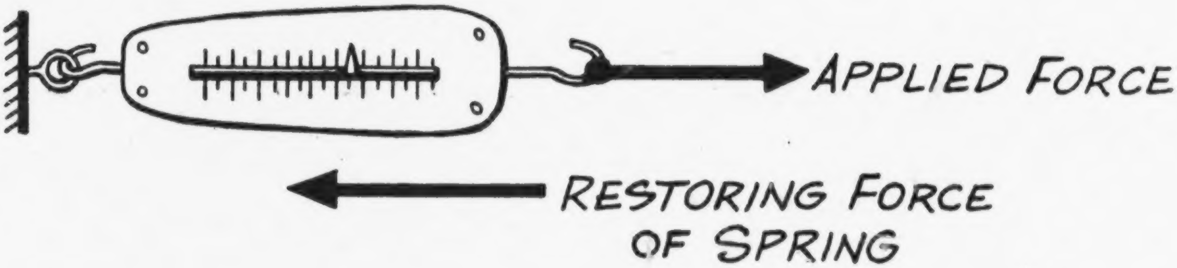

APPLIED FORCE
RESTORING FORCE OF SPRING

FIGURE ③

WORK DONE IN COMPRESSING SPRING

$$= \int F\,dx = \int_0^x kx\,dx$$

$$= \frac{1}{2}kx^2$$

FIGURE ④

POTENTIAL ENERGY

$$= \frac{1}{2}\ kx^2$$

FIGURE ⑤

Is it possible to set up and analyze a composite situation, that is, one in which the work done is partially converted into a change in kinetic energy and also partially into a change in potential energy? The answer is - yes, it can be done quite simply for the conditions shown in Figure 6. A mass, securely fastened to the end of a horizontal spring, is acted on by a force $\vec{F}_a$ to the right. Two things happen simultaneously: the mass goes into motion, gaining kinetic energy, and the spring begins to exhibit compression. Suppose that the mass is displaced a distance x in the process as indicated in Figure 7. This compression gives rise to a restoring force equal to the spring constant k multiplied by the compression x. The resultant unbalanced force on the mass must therefore be the difference between the applied force and the restoring force or $F_a - kx$. This is summarized in Figure 8. The student should pause at this point and contemplate the implications of the development thus far.

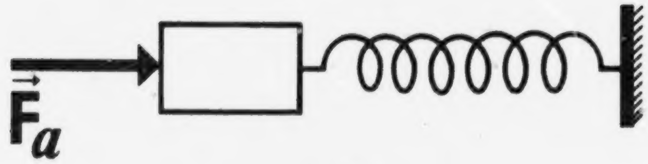


FIGURE ⑥

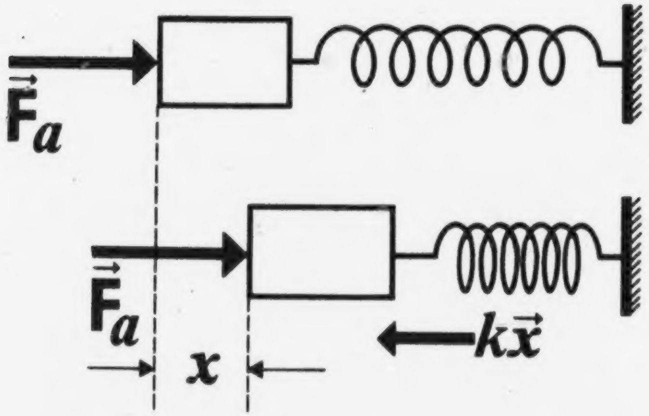
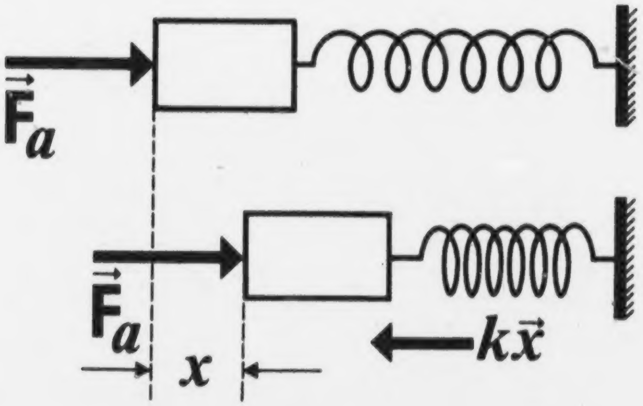

$\vec{F}_a$
$\vec{F}_a$
$k\vec{x}$
x

FIGURE ⑦

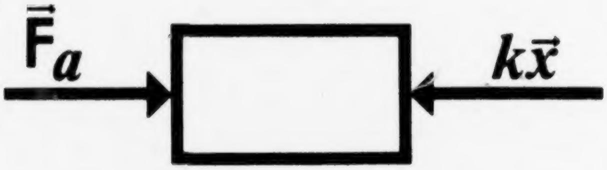

$\vec{F}_a$
$\vec{F}_a$
$k\vec{x}$
x

RESULTANT $= F_a - kx$

FIGURE ⑧

A free-body diagram is next in view in Figure 9. The mass on the end of the spring is acted upon by $\vec{F}_a$, the applied force, to the right and by the restoring force $k\vec{x}$ to the left. The difference between these two forces is the resultant force on the mass. The external work done by the agency that supplies the force $\vec{F}_a$ is the integral of F dx as before; in compressing the spring, this agency contributes to the potential energy of the spring, this potential energy being the integral of kx dx, of course. This is summarized in Figure 10.

What, then, is the action of the resultant force $F_a - kx$? The kinetic energy of the system must change as a result of this action. Mathematically expressed, the situation may be described as shown in Figure 11. Descriptively, this means that the external work done by the agency that applies the force $\vec{F}_a$ is converted into both potential energy of compression and also into a change of kinetic energy of the mass.



Free Body Diagram

FIGURE ⑨

WORK DONE

$$= \int F \, dx$$

$$\Delta \, P.E. = \int_0^x kx \, dx$$

FIGURE ⑩

WORK DONE

$$= \int F \, dx$$

$$\Delta \, P.E. = \int_0^x kx \, dx$$

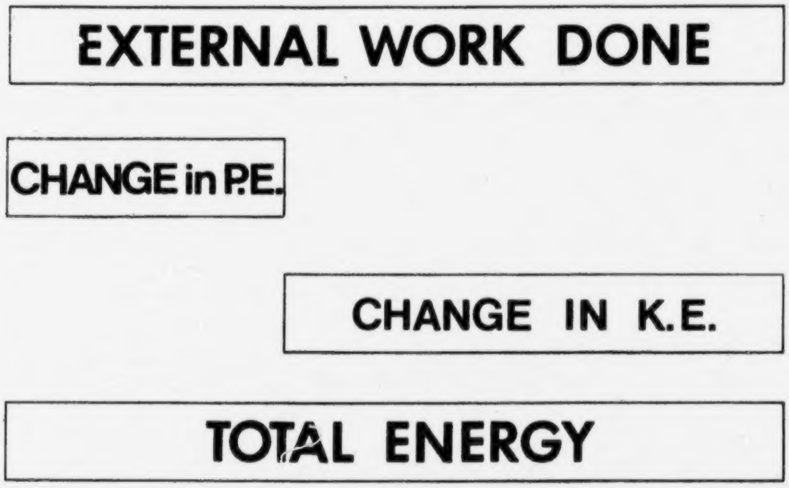
$$\Delta \, K.E. = \int_0^x (F_a - kx) \, dx$$

FIGURE ⑪

The "bar graphs" in Figure 12 express this result graphically. The sum of the energy changes in the system must be equal to the total energy change in the system. In the most general terms, when external work is done on a body in any system there may be a change in the potential energy of the system or in the kinetic energy of the system *or in both*. In any event, regardless of the alternatives that are followed, the external work done must equal the total energy change of the system.

Figure 13 indicates a state of affairs which may at first appear trivial but which most assuredly is not. If no external work is done on a system, the change in total energy in the system is also zero. *But this does not mean that neither the potential energy nor the kinetic energy has changed.* It merely means that, whatever changes do occur when the work done on the system is zero, these changes must compensate for one another. Refer to Figure 14. If there is a positive change (increase) in kinetic energy, then there will be an equal negative change (decrease) in potential energy if the work done is zero.

This is the essence of the Law of Conservation of Energy. In part, it states that the total work done on a system must be equal to the algebraic sum of the enery changes that occur in the system as a result of this work. This is often called the "work-energy theorem". A second implication is that, even when no work is done on a system by an outside agency, there may still be changes in potential and kinetic energy but that these changes are compensatory. What is gained in one form is lost in another. Energy cannot be created nor destroyed but only changed in form.



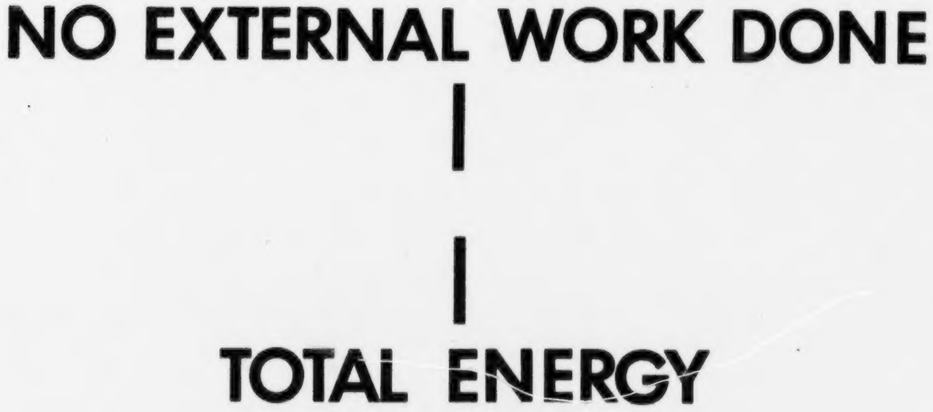
EXTERNAL WORK DONE

CHANGE in P.E.

CHANGE IN K.E.

TOTAL ENERGY

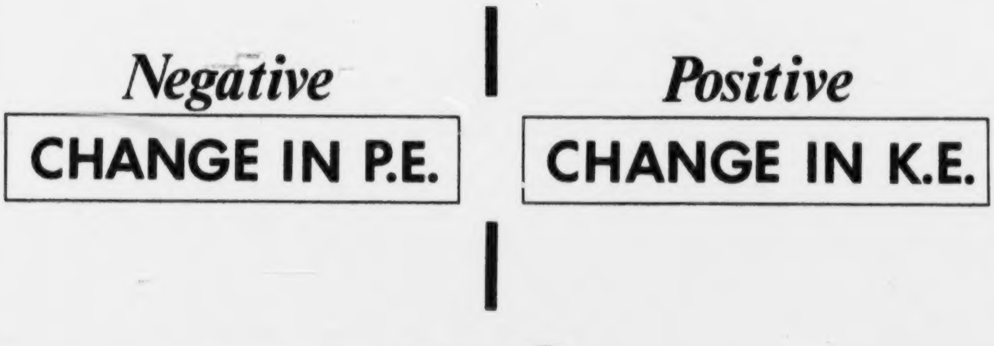
FIGURE ⑫



NO EXTERNAL WORK DONE

TOTAL ENERGY

FIGURE ⑬



Negative

CHANGE IN P.E.

Positive

CHANGE IN K.E.

FIGURE ⑭

CONSERVATION
OF
ENERGY

TERMINAL OBJECTIVES

5/2 C	Answer questions pertaining to the statement of conservation of energy.
5/3 B	Apply conservation of energy to a simple pendulum.
5/3 C	Demonstrate a knowledge of specifies required for the application of the Conservation of Energy theorem.

Please turn to page 34A of your STUDY GUIDE to continue with your work.

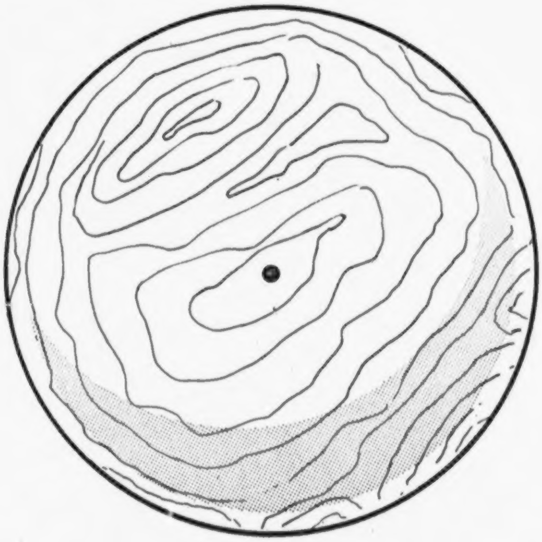
MOVEMENT OF CENTER OF MASS

The center of mass of an object may be described as that single point at which all of its mass appears to act. For an object of uniform density having some regular shape, such as a solid wooden ball, its center of mass is easily located to be at the geometric center, as you can see in Figure 1. Finding the location of the center of mass for a hollow rubber ball is no more difficult--it too is at the geometric center, even though none of the actual mass of the ball is located at that very point.

Many objects, having either regular or irregular shapes, have centers of mass located in space--probably the chair you are sitting on at this moment or the cup or glass you used this morning are good examples to consider. For these objects, the center of mass acts in every way just as it does for one having a center of mass within the medium itself--as with the solid ball.

CENTER OF MASS

(a) for a solid ball



(b) for a hollow ball

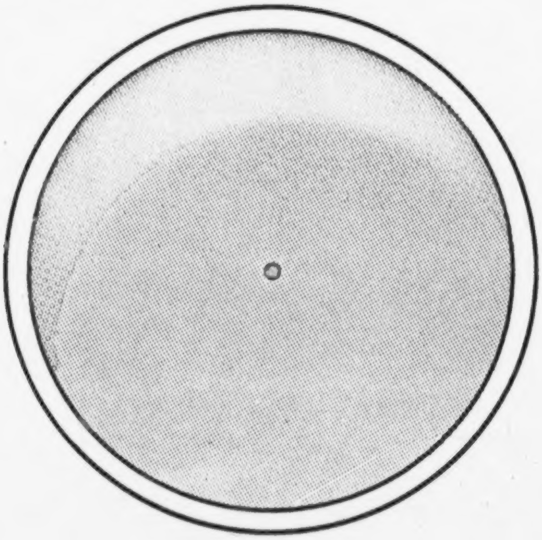


FIGURE ①

The concept of center of mass can be a powerful tool in the study of motion, since all rigid bodies, regardless of shape, volume, or density, can be considered to be point masses acted upon by external forces, thereby simplifying the application of Newton's laws of motion.

A task that at first seems difficult is the analysis of the motion of a body when internal forces are also acting. Let's see what effect, if any, they might have. To do this, let's examine the effect of an explosion on the center of mass of a system consisting of two equal masses. In Figure 2, you see two identical cars about to be exploded apart by a compressed spring. Before the explosion, the center of mass of the system is midway between the cars. When the explosion occurs, each car receives an equal, but opposite force to the other, for the same period of time, giving each similar accelerations. But at any time, the center of mass of the system can be found to be at the same point, unaffected by the explosion.

EQUAL MASS CARS

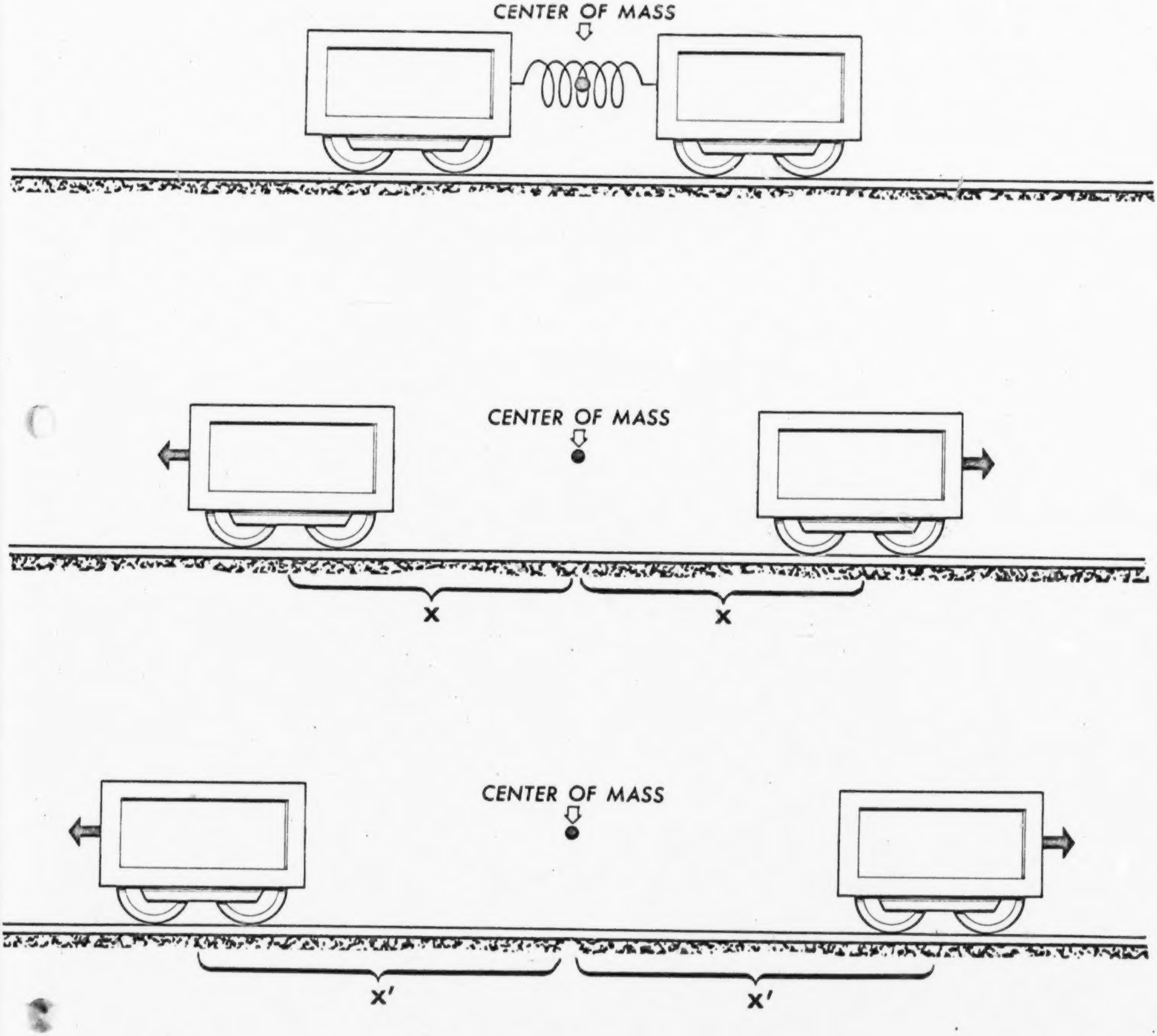

CENTER OF MASS
CENTER OF MASS
x
x
CENTER OF MASS
x'
x'

FIGURE ②

You may well ask, what would have happened if two unequal masses were chosen? Let's repeat the explosion, this time with unequal cars; say they have a mass ratio between them of 1:2. Once again the explosion will apply equal and opposite forces on the cars, but this time one car, the lighter one, will accelerate at twice that of the heavy car, thereby moving twice as far in equal time. Consequently, the center of mass of the system remains in the same position, unaffected by internal forces as you can see by examining Figure 3. As a matter of fact, even if the two cars have some initial velocity while linked together, their center of mass would continue to move at that velocity even after the explosion occurs.

UNEQUAL MASS CARS

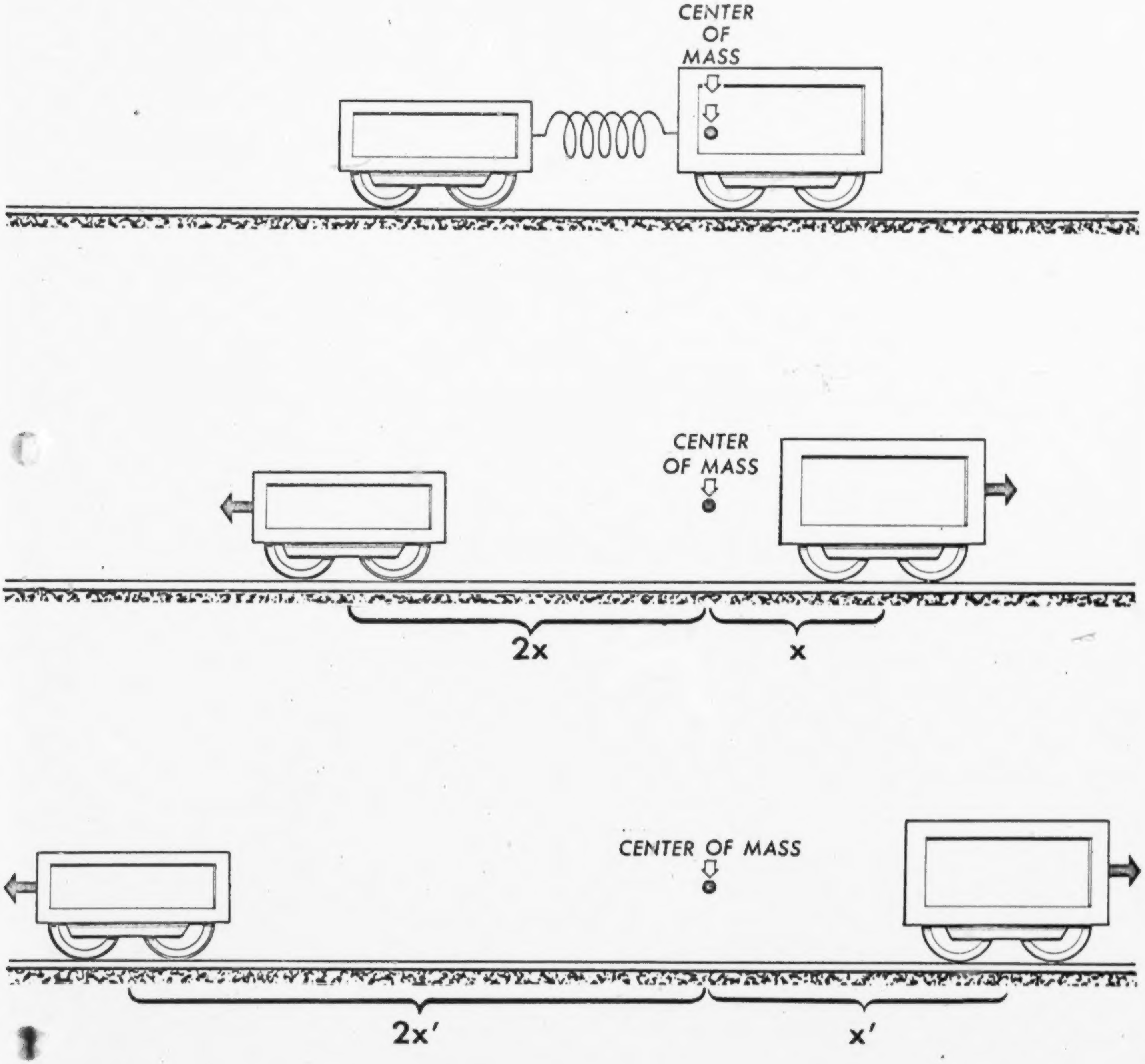

CENTER OF MASS
CENTER OF MASS
2x
x
CENTER OF MASS
2x′
x′

FIGURE ③

Before closing, let's apply these principles to some typical motion problem. A good one to consider would be the motion of an explodable ball as it moves in a parabolic trajectory. Here, in Figure 4, the ball is subjected to some initial accelerating force, and a constant gravitation force, both acting externally, as well as an internal explosive force.

Before the explosion the ball travels intact along a parabolic path governed by the effects of its initial velocity and gravitation. The ball is then exploded into fragments, each moving away from the center of gravity at a rate dependent upon the explosive force and its size, and each still is affected by the initial velocity and gravitation. Since the explosive internal force has been shown to have no effect on the center of gravity, its motion continues along the parabolic trajectory as though the ball had remained intact.

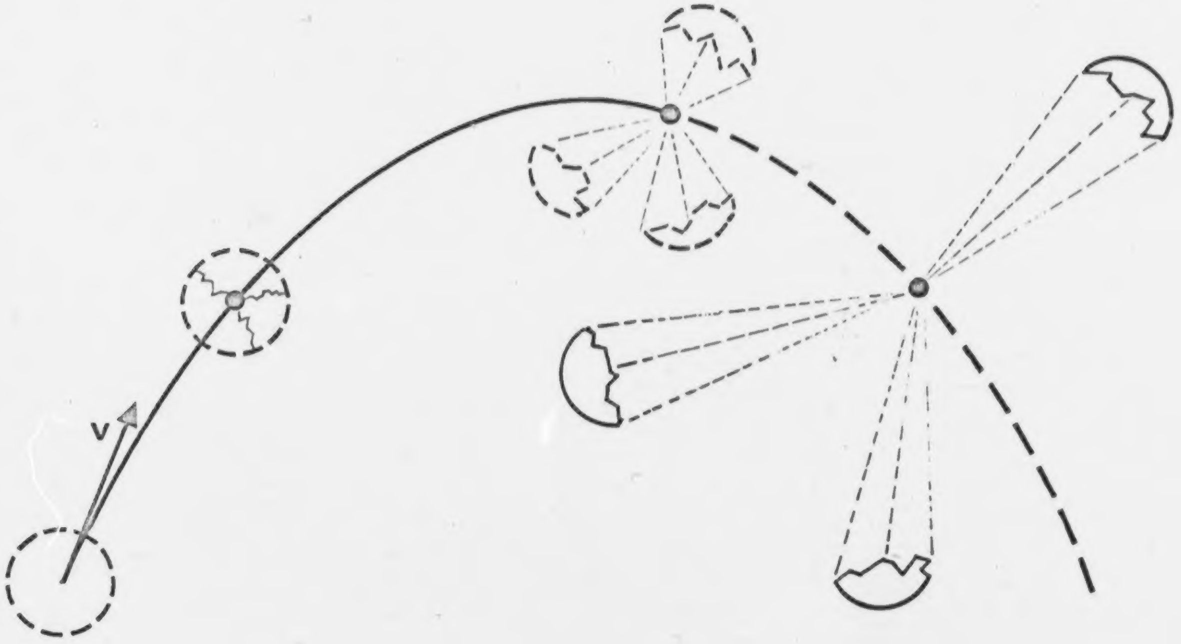


FIGURE ④

CONSERVATION OF MOMENTUM

As a fundamental physical principle, conservation of momentum ranks among the most important; in many situations its usefulness exceeds that of even the principle of conservation of energy. This is especially true of collisions between moving objects because there is little about such collisions that cannot be predicted or explained with the aid of the principle of conservation of momentum.

Figure 1 diagrammatically depicts two bodies, A and B, the first having a mass of m_A and the other a mass of m_B. Body A is moving toward the right with a velocity $\vec{v}_A$ while body B, also moving from left to right, has a velocity $\vec{v}_B$. Velocity v_A is larger in magnitude than velocity v_B as indicated by the relative lengths of the vector arrows for each quantity. Given sufficient time, body A will close the separation between the two and will eventually collide with body B. This event is illustrated in Figure 2.

Assuming that the bodies do not adhere to one another, they will separate after the collision and move off with velocities that will in most cases differ from the initial values. The velocities subsequent to collision are symbolized in the diagram shown in Figure 3 as $\vec{v}_A$, and $\vec{v}_B$, respectively.

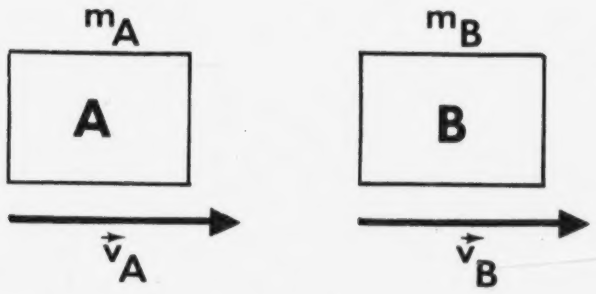

m_A
A
$\vec{v}_A$
m_B
B
$\vec{v}_B$

FIGURE ①

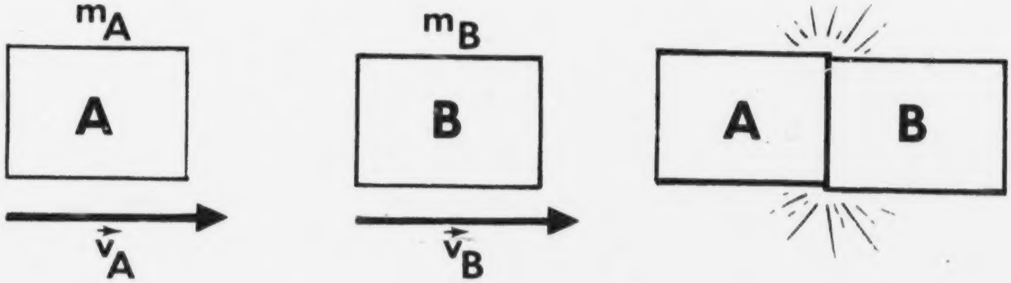

m_A
A
$\vec{v}_A$
m_B
B
$\vec{v}_B$
A
B

FIGURE ②

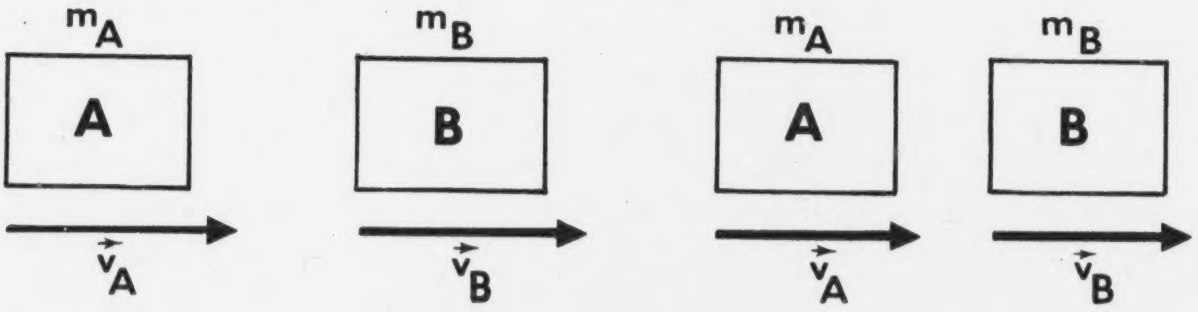

m_A
A
$\vec{v}_A$
m_B
B
$\vec{v}_B$
m_A
A
$\vec{v}_A$
m_B
B
$\vec{v}_B$

FIGURE ③

Regardless of the nature of the collision, that is whether it occurs between elastic or inelastic bodies or between hard or soft bodies, the objects will be in contact with one another for a definite time interval. In Figure 4, the force that body A exerts on body B -- $\vec{F}_{AB}$ -- is plotted against time. At the instant that body A overtakes body B and first contact is established, the force will start to rise from zero. About half-way through the contact interval, the force will have risen to a peak and then, as separation or rebound begins, the force will diminish until it vanishes entirely as separation becomes complete. A collision, therefore, involves a varying force acting over a definite time.

For a constant force, the impulse is given as $\vec{F}\Delta t$. When the force varies from instant to instant as it does in this example, the impulse can be determined most easily by integrating all the $F\Delta t$ products under the curve. Thus, as indicated in Figure 4, the impulse is the integral of $\vec{F}.dt$ between the limits extending from zero time (first contact) to the time when separation is completed. This integral is the equivalent of the area under the curve in Figure 4.

Since this interaction is typical of the kind of phenomenon to which Newton's Third Law rigorously applies, it is possible to state immediately that the force exerted on body A by body B is identical in instantaneous magnitude with $\vec{F}_{AB}$ throughout the interval but, of course, is oppositely directed. Furthermore, since the time of interaction is the same for $\vec{F}_{AB}$ and the reaction force, then the impulse of B on A must equal the impulse of A on B. This is shown graphically in Figure 5, and a verbal statement is given in Figure 6. The negative sign is inserted on the right side of the statement because this is a convenient way to indicate the "oppositeness" of direction.

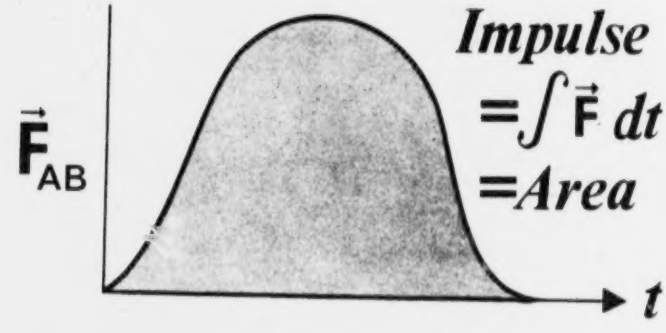


FIGURE ④

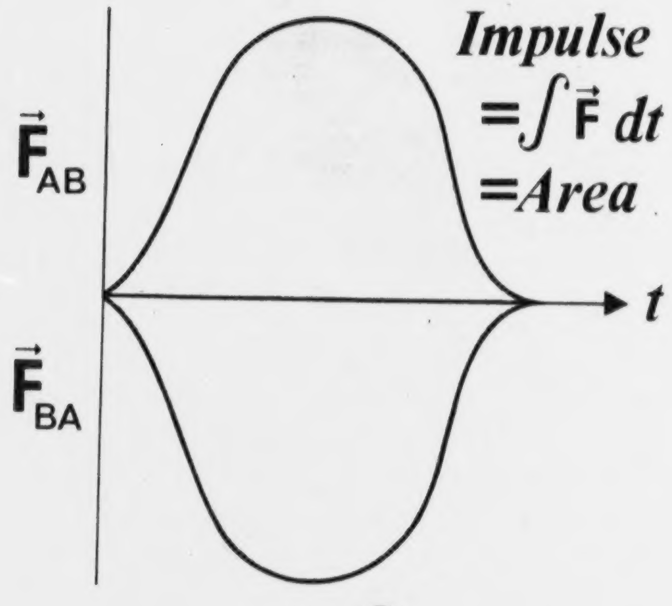


FIGURE ⑤

Impulse applied to A = −Impulse applied to B

FIGURE ⑥

The impulse acting on a body is equal to its change of momentum. Hence, the statement of Figure 6 is equally valid when written as shown in Figure 7. The impulse applied to B becomes the change in momentum of B; the impulse on A becomes the (negative) change in momentum of A. Alternatively, whatever momentum is acquired by B is lost by A, or vice versa. In any of these statements, the negative sign may be shifted from one side to the other without altering the significance of the statement. Finally, as in Figure 8, the implication may be succinctly stated: there is no change in momentum or momentum is conserved.

$$\left.\begin{matrix}\textit{Impulse}\\ \textit{applied to } A\end{matrix}\right\} = \left\{\begin{matrix}-\textit{Impulse}\\ \textit{applied to } B\end{matrix}\right.$$

momentum change of A
= −momentum change of B

FIGURE ⑦

$$\left.\begin{matrix}\textit{Impulse}\\ \textit{applied to } A\end{matrix}\right\} = \left\{\begin{matrix}-\textit{Impulse}\\ \textit{applied to } B\end{matrix}\right.$$

momentum change of A
= −momentum change of B

No change in momentum

FIGURE ⑧

In the preceding development, no attempt was made to specify the nature of the colliding bodies, their masses, their initial velocities, or the nature of the collision -- whether elastic or inelastic. The application of the third law to any type of collision between any two bodies confirms that momentum is always conserved. (It should be noted here that this discussion has been limited to head-on collisions and that the third law, as applied, was also limited in this respect. It will be shown, however, that this limitation is not required; the collision may be of any variety -- momentum is still conserved.) The principle of conservation of momentum is one of the ultimate truths of Nature.

The general approach to the solution of conservation of momentum problems is quite straightforward. Starting with the conditions shown in Figure 9, the sequence of two colliding bodies before impact, during impact, and soon after impact, one may write a relationship that expresses the conservation principle in a step-by-step procedure like this:

1. Write the total momentum of the system before collision as the sum of the individual momenta of the bodies as

$$m_A\vec{v}_A + m_B\vec{v}_B$$

To be strictly correct, the velocities should be indicated as vectors.

2. Write the total momentum of the system after collision as the sum of the individual momenta of the bodies after the interaction has occurred as:

$$m_A\vec{v}_{A'} + m_B\vec{v}_{B'}$$

3. And finally since the sums must be equal before and after collision, equate the two sums as shown in Figure 10. When using this single equation in problem work, only one unknown, of course, is permissible.

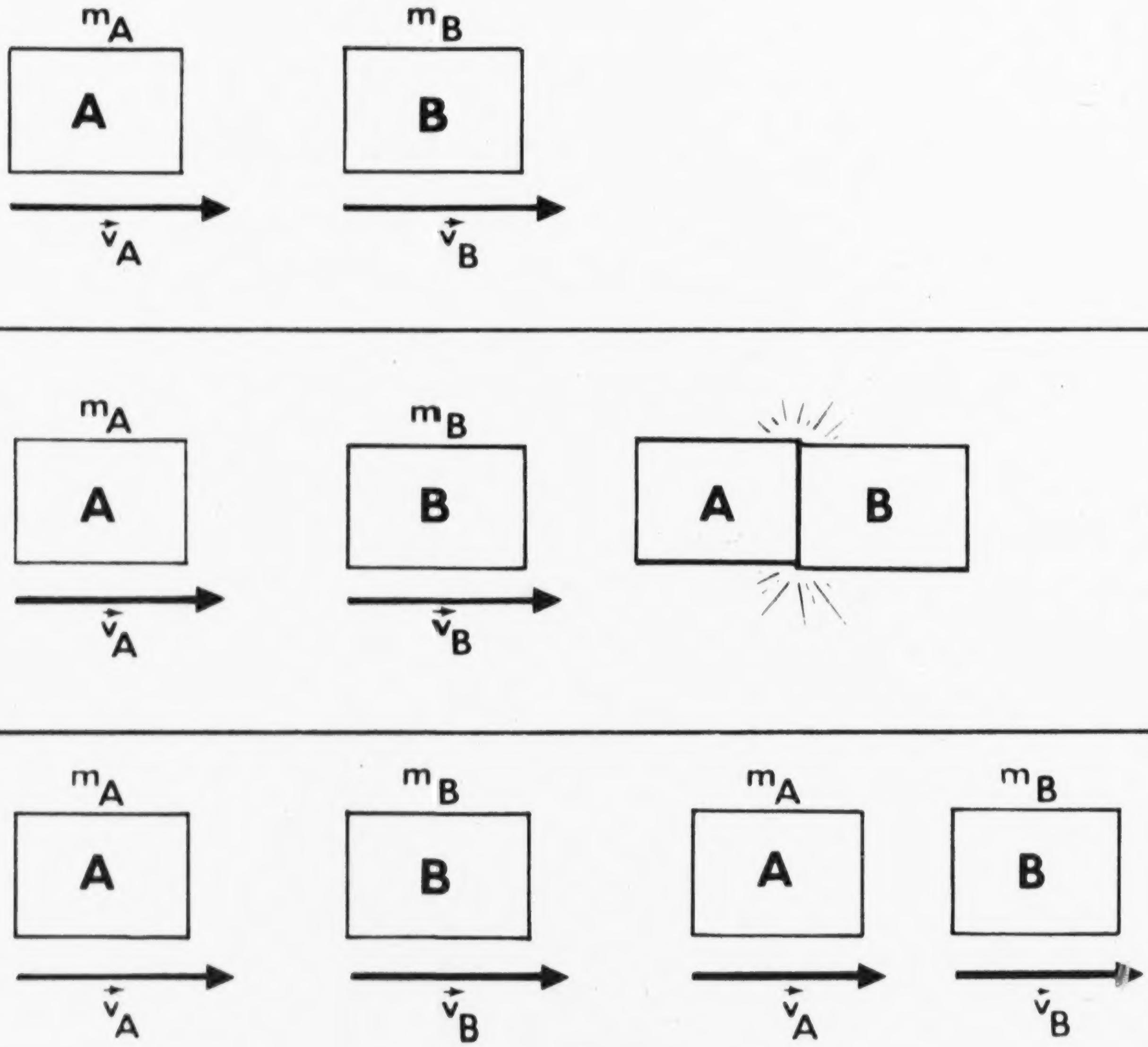

m_A
m_B
A
B
$\vec{v}_A$
$\vec{v}_B$
m_A
m_B
A
B
A
B
$\vec{v}_A$
$\vec{v}_B$
m_A
m_B
m_A
m_B
A
B
A
B
$\vec{v}_A$
$\vec{v}_B$
$\vec{v}_A$
$\vec{v}_B$

FIGURE ⑨

$$\underbrace{m_A \vec{v}_A + m_B \vec{v}_B}_{\text{MOMENTUM BEFORE COLLISION}} = \underbrace{m_A \vec{v}_{A'} + m_B \vec{v}_{B'}}_{\text{MOMENTUM AFTER COLLISION}}$$

FIGURE ⑩

CONSERVATION OF MOMENTUM

TERMINAL OBJECTIVES

6/2 B	Solve momentum problems involving bodies with variable mass.
6/2 C	Analyze situations and phenomena in which momentum is a significant factor.

Please turn to Page 21A of your STUDY GUIDE to continue with your work.

IMPULSE
AND
MOMENTUM

The purpose of this exposition is to define the terms *impulse* and *momentum* as they are used in physics, and establish the relationship between them.

As a brief review, it is perhaps wise to reexamine the connection between force, displacement, and work for the event illustrated in Figure 1. A force is applied to a body resting on a horizontal, frictionless table. As a result of the application of the unbalanced force $\vec{F}$, the body is displaced through a distance $\underline{\vec{x}}$. The work done on the body, $\vec{F}\cdot\vec{x}$ is thereby converted to the energy of motion or kinetic energy and, in accordance with the Work-Energy Theorem, the work done on the body is equal to the *change* of kinetic energy that the body undergoes.

A somewhat different aspect of this event involves the measurement of time to establish the interval over which the motion takes place. See Figure 2. In this approach, the displacement is ignored; the time interval from one position to the next denotes the *interval over which the force acts on the body*.

$$\vec{F} \cdot \vec{x} = \Delta KE$$

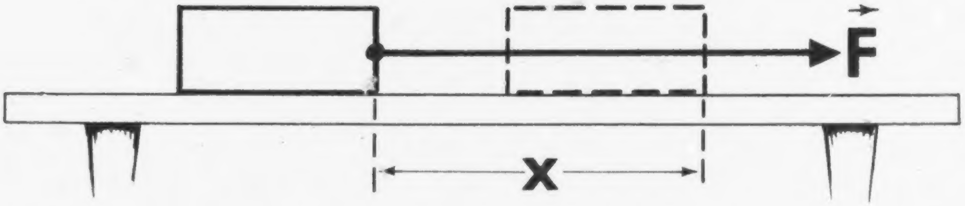


FIGURE ①

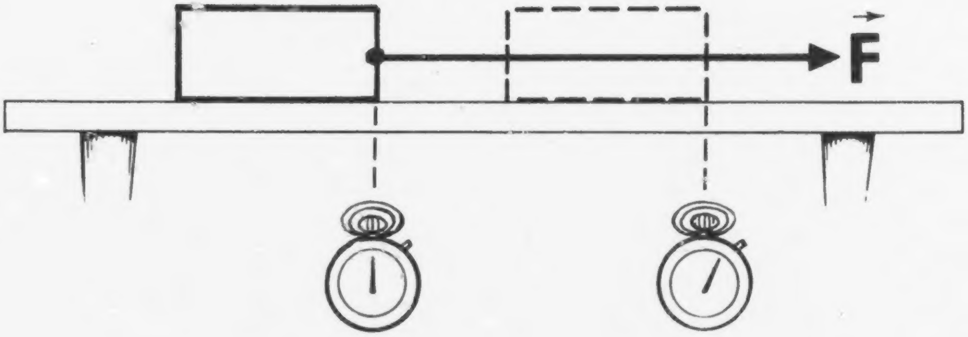


FIGURE ②

From the analytical point of view, a rather direct attack can be made on the problem by using Newton's Second Law. As in Figure 3, the second law may be written by vector form as a relationship between force, mass, and acceleration. Then acceleration is redefined as $d\vec{v}/dt$ for convenience and substituted for it in the expression as shown.

Assuming next that the force is to be applied for a short time interval dt, as illustrated in Figure 4, the product $\vec{F}\cdot dt$ is formed on the left side making it necessary to multiply the right side by dt to maintain the equality. The dt's then drop out leaving the expression given in Figure 5.

Clearly the equation is in vector form because the quantity $d\vec{v}$ is a vector. In dealing with work and energy, the concepts obviously lead to a scalar equation; in the development of impulse and momentum it is just as obvious that vector equations will appear.

$$\vec{F} = m\vec{a}$$

$$\vec{F} = m \frac{d\vec{v}}{dt}$$

FIGURE ③

$$\vec{F} = m\vec{a}$$

$$\vec{F} = m \frac{d\vec{v}}{dt}$$

$$\vec{F}\,dt = m \frac{d\vec{v}}{\cancel{dt}} \cancel{dt} = m\,d\vec{v}$$

FIGURE ④

$$\vec{F} = m\vec{a}$$

$$\vec{F} = m \frac{d\vec{v}}{dt}$$

$$\vec{F}\,dt = m \frac{d\vec{v}}{\cancel{dt}} \cancel{dt} = m\,d\vec{v}$$

$$\vec{F}\,dt = m\,d\vec{v}$$

(vector equation)

FIGURE ⑤

To find the resultant effect of applying an unbalanced force over a given time, it is necessary to integrate $\vec{F} \cdot dt$ over this interval. The left member of the expression in Figure 6 shows how this is written. In the same equation, the right member has been written to show that the change in momentum must be integrated between the limits v_1 and v_2. These velocities represent the range of variation of the velocity of the body over the time interval t_1 to t_2, of course.

In order to integrate the left member, the precise dependence of force on time must be known and, since it is not known, this member is left as an unperformed integral. The right member, however, may be readily integrated in its present form.

This operation is given in Figure 7. The integral of $m.d\vec{v}$ between the limits v_1 and v_2 is simply mv_2-mv_1 as shown. The student is urged to verify this before proceeding.

The right member now expresses a change of momentum; the difference between the initial momentum mv_1 and the final momentum mv_2. The integral of $\vec{F}$ dt from t_1 to t_2 is called the impulse of the force. The equation as it now stands is a concise mathematical statement of what has come to be called the impulse-momentum theorem. Figure 8, then, presents the complete sequence which terminates in the impulse-momentum theorem: IMPULSE = CHANGE IN MOMENTUM. Expanding on this somewhat, it can be restated that the impulse of an unbalanced force applied to a body is always equal to the change in momentum that this impulse produces.

$$\vec{F}\,dt = m\,d\vec{v}$$

$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{v_1}^{v_2} m\,d\vec{v}$$

FIGURE ⑥

$$\vec{F}\,dt = m\,d\vec{v}$$

$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{v_1}^{v_2} m\,d\vec{v}$$

$$\int_{t_1}^{t_2} \vec{F}\,dt = m\vec{v}_2 - m\vec{v}_1$$

FIGURE ⑦

$$\vec{F}\,dt = m\,d\vec{v}$$

$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{v_1}^{v_2} m\,d\vec{v}$$

$$\int_{t_1}^{t_2} \vec{F}\,dt = m\vec{v}_2 - m\vec{v}_1$$

Impulse = *change in momentum*

FIGURE ⑧

It would be fruitful to work on a specific problem dealing with impulse and momentum at this juncture. In Figure 9 is depicted an ordinary carpenter's hammer striking a nail which is to be driven into a block of wood. Although most people intuitively understand why this process can be successfully performed, an analytical approach to this problem is not difficult and can be quite illuminating. Figure 10 presents some reasonable figures for the mass or weight of the hammer and its impact velocity in the hands of an ordinary man.

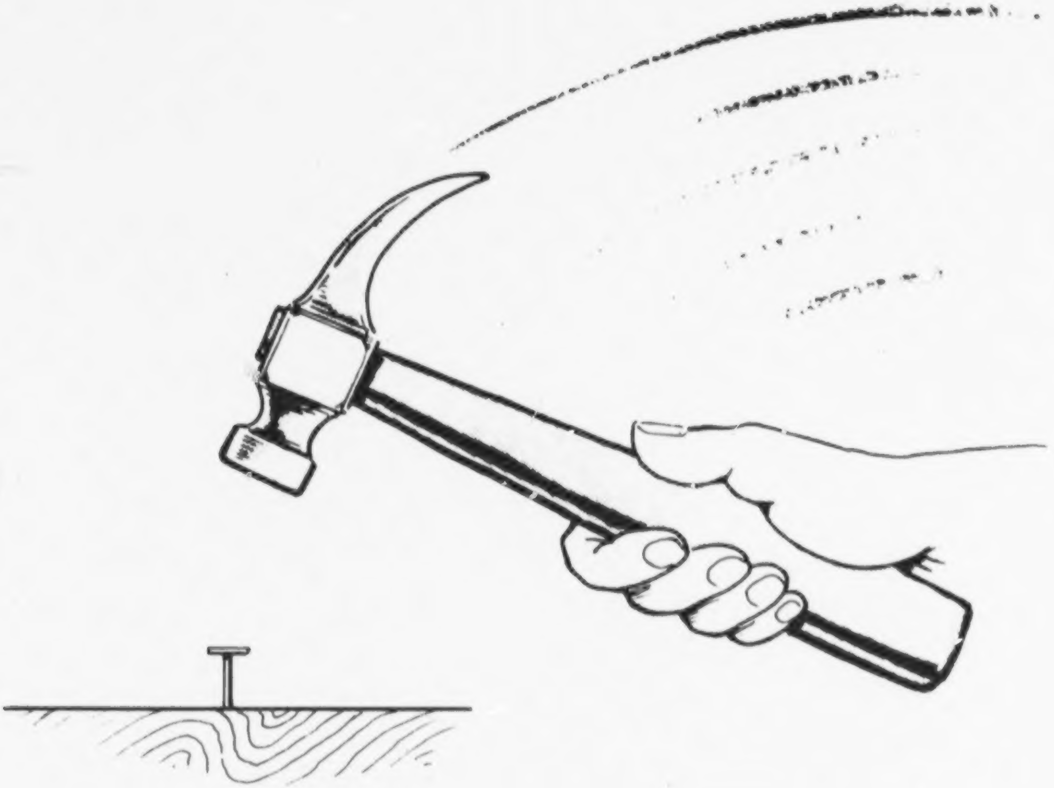


FIGURE ⑨

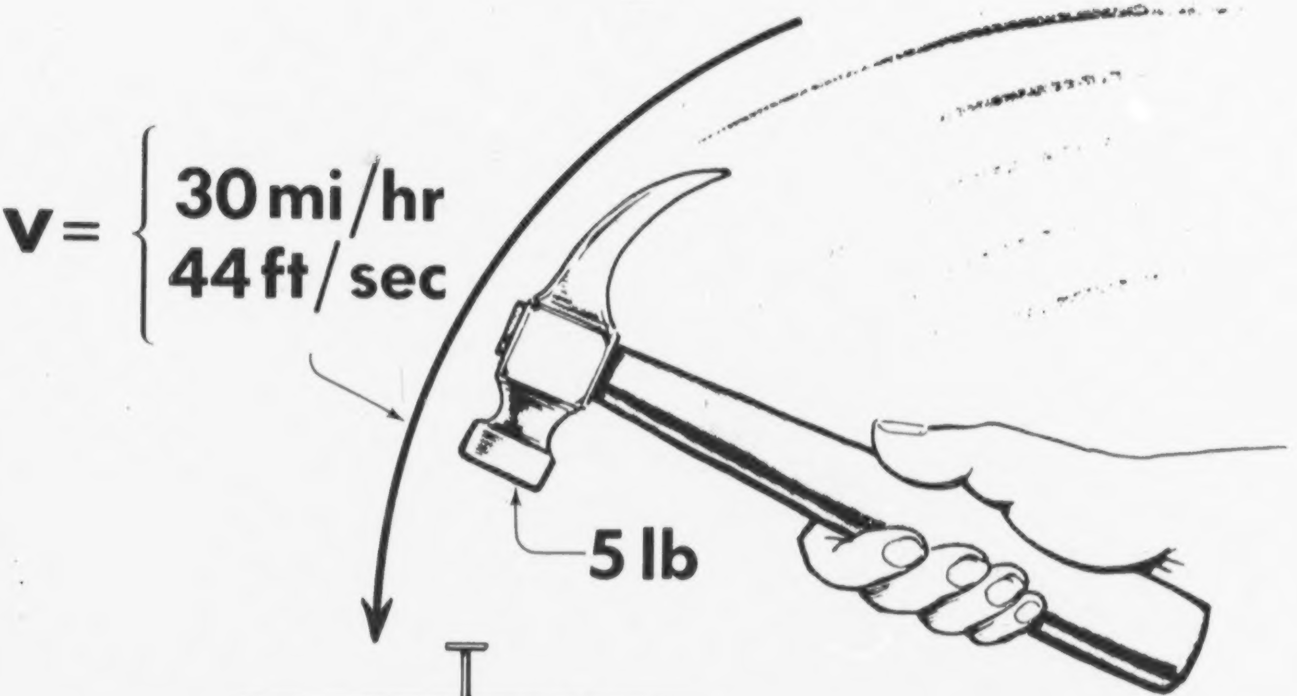

V = 30 mi/hr
44 ft/sec
5 lb

FIGURE ⑩

Assuming that the hammer comes to rest after striking the nail, what force does it exert during impact? This problem lends itself to solution by the impulse-momentum theorem. The information in Figure 11 should now be reviewed. First there is a statement of the impulse-momentum theorem; second, the weight of the hammer -- 5 lb. -- is converted into mass by dividing the weight by g. This is derived from the second law equation $w = mg$, so $m = w/g$; third, the initial velocity of the hammer is given as v_1; and fourth, the final velocity of the hammer is given as zero since the hammer is assumed to stop moving upon impact.

Figure 12 contains the statement that the change in momentum, or $mv_2 - mv_1$, is 5/32 slug multiplied by 44 ft/sec, using the given data. The next requirement is to find out something about the impulse of the hammer on the head of the nail. Figure 13 gives these details: impulse is the product of force and time but in this case it must be recognized that the hammer applies some average force to the nail through the interval of contact. Assuming that the hammer remains in contact with the nail for 1/100 second, then the impulse is the average force $\bar{F} \times 1/100$.

$$\text{Impulse} = \begin{matrix}\text{change in}\\ \text{momentum}\end{matrix}$$

$$\text{Mass} = \frac{5}{32}\ \text{slug}$$

$$v_1 = 44\ {}^{ft}/_{sec}$$

$$v_2 = 0\ {}^{ft}/_{sec}$$

FIGURE ⑪

$$\begin{matrix}\text{change in}\\ \text{momentum}\end{matrix} = \frac{5}{32} \times 44 - 0\ \frac{\text{slug ft}}{\text{sec}}$$

FIGURE ⑫

$$\begin{matrix}\text{change in}\\ \text{momentum}\end{matrix} = \frac{5}{32} \times 44 - 0\ \frac{\text{slug ft}}{\text{sec}}$$

$$\text{impulse} = \bar{F}t \qquad \bar{F} = \begin{matrix}\text{average}\\ \text{force}\end{matrix}$$

$$= \bar{F}\,\frac{1}{100}$$

FIGURE ⑬

The last sequence of steps in the solution of the problem appears in Figure 14. In the first step, impulse ($\bar{F}$ x 1/100) is set equal to the change in momentum (5/32 x 44). Solving for the average force $\bar{F}$, the result is approximately 687 lb. or 1/3 ton.

The student should consider how a 5-lb. hammer can exert so large an average force on the nail. A little thought should show that this large force is obtained by giving the hammer a large momentum through the medium of a very large impact velocity. Then, since the contact time is so short, the resulting large impulse must yield a correspondingly large force.

$$\bar{F} \times \frac{1}{100} = \frac{5}{32} \times 44$$

$$\bar{F} = \left(100 \times \frac{5}{32} \times 44 \right) \text{lbs}$$

$$= 687 \text{ lbs or } \frac{1}{3} \text{ ton (APPROX)}$$

FIGURE ⑭

IMPULSE
AND
MOMENTUM

TERMINAL OBJECTIVES

6/2 A	Solve momentum problems involving bodies with constant mass.
6/3 A	Analyze situations which involve net impulsive forces acting on bodies of constant mass.

Please turn to page 31A of your STUDY GUIDE to continue with your work.

COLLISIONS

Collisions

In an isolated system involving two or more bodies which interact with one another, the momentum at any instant is the same as it is at every other instant regardless of the number of kinds of interactions that occur. An isolated system is one in which no external forces act to change the momentum of any of the bodies within the system. Essentially, this is a statement of the principle of conservation of momentum. From a purely theoretical point of view, it is readily seen why momentum must be conserved. Selecting a simple case, that of a collision between two bodies in an isolated system, the forces that act on each body during the collision must be equal and opposite (Third Law) and, since the time of impact is also the same, then equal impulses act on both bodies. Impulse is equal to change of momentum, hence the change of momentum of each body involved in the collision must also be the same. It should be noted that the kind of collision that occurs --- elastic, inelastic, or a combination --- does not affect the validity of the momentum conservation principle.

Kinetic energy on the other hand is not necessarily conserved in all collisions. Normally a collision is accompanied by the development of sound and heat; these are lost to the system so that the total energy content after the collision must be less than it was initially. A collision in which kinetic energy is conserved may be closely approximated, however, with the proper kind of apparatus. Such a collision is termed perfectly elastic. At the other extreme in which the kinetic energy content of the system is zero after the collision is the perfectly inelastic collision. Most real collisions are partly elastic and partly inelastic. To study a close approach to a perfectly elastic collision, an air track is usually utilized.

(Figure 1) This piece of equipment consists of a hollow, triangular cross-section rail that may be several meters long. Air from a compressor is forced into the hollow section and emerges from a large number of very fine holes in the sloping sides. A close-fitting glider when placed on the rail is lifted very slightly so that it rides on a thin layer of air. The friction is thereby reduced to a negligible value. When two such gliders, equipped with flexible, soft springs, are allowed to collide, it is found that kinetic energy is essentially conserved, hence the collision is very nearly perfectly elastic. In the sample shown in the figure, there are two gliders of exactly equal mass m on the rail. For simplicity, it is assumed that glider 2 is at rest while glider 1 is set in motion toward it from left to right with a velocity u_1.

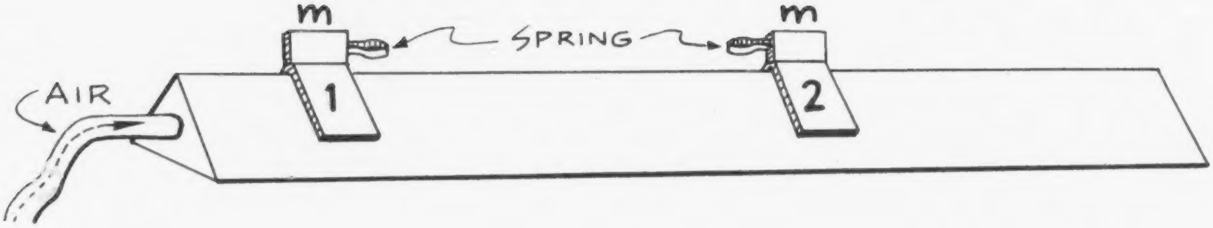

m
m
SPRING
AIR
1
2

FIGURE ①

(Figure 2) These are the symbols to be used in this discussion.

Since glider 2 is at rest, its velocity before collision u_2 is taken as zero. The velocities of glider 1 and glider 2 after the collision are respectively v_1 and v_2.

(Figure 3) To write the equation that expresses conservation of momentum for the type of collision described, it is first necessary to write the total momentum of the system before the collision. As shown in Figure 3, the total system momentum before the collision is mu_1, that is the product of the mass of glider 1 and its velocity. Since glider 2 was initially at rest, it has no momentum and hence need not appear in the terms in front of the equals sign to be written in the equation. The system momentum after the collision is $mv_1 + mv_2$, the sum of the momenta of the individual gliders. Note that the assumption is made that glider 2 is set in motion as a result of the impact with a velocity v_2 and that the velocity of glider 1 changes from u_1 to v_1, also as a result of the collision.

(Figure 4) The first equation in Figure 4 is merely a statement of the fact that momentum is conserved since the total momentum before the collision has been equated with the total momentum after the collision. Since the gliders have the same mass, the factor m is the same for all terms and may be eliminated by dividing through as shown in the second equation. Verbally, the second equation states that the algebraic sum of the velocities after the collision is equal to the velocity of glider 1 before the collision. The next thing to be considered are the kinetic energies of the gliders before and after the collision.

$m = m$ = mass of each glider

u_1 = velocity of glider 1 BEFORE collision

u_2 = velocity of glider 2 BEFORE collision = 0

v_1 = velocity of glider 1 AFTER collision

v_2 = velocity of glider 2 AFTER collision

FIGURE ②

Before collision the system momentum = mu_1

After the collision, the system momentum = $mv_1 + mv_2$

FIGURE ③

$mu_1 = mv_1 + mv_2$ and dividing by m

$u_1 = v_1 + v_2$

FIGURE ④

(Figure 5) In general, the kinetic energy of a moving mass m having a velocity v is 1/2 mv^2. Thus, the kinetic energy of glider 1 before the collision is 1/2 mu_1^2 and the total kinetic energy after the collision is 1/2 mv_1^2 + 1/2 mv_2^2. If the collision is perfectly elastic, then kinetic energy is conserved so that it may be expressed as shown in the top equation. Here again, m is the same throughout and may be eliminated to yield the second equation.

(Figure 6) These are the two final expressions previously obtained, the first from the principle of conservation of momentum and the second from the principle of conservation of kinetic energy, both based upon the same collision. This point cannot be overemphasized. Since both equations are perfectly valid, it must be concluded that the change of velocity of each body in an elastic collision must be such as to satisfy two separate conditions simultaneously: (1) the sum of the final velocities must equal the initial velocity and (2) the square of the initial velocity must equal the sum of the squares of the final velocities.

(Figure 7) The implication of this double-barreled requirement is most easily seen by combining the two equations as shown here. The linear equation is first squared and then one equation is subtracted from the other. The result is obtained that twice the product of the final velocities must be equal to zero. This further implies that any one of the following possibilities may have occurred:

(1) Possibly $v_1 = 0$. This would be the case only if glider 1 stopped in its tracks immediately upon impact. When the experiment is perfored it is found that this is indeed the case: glider 1 stops dead while glider 2 goes off with the same velocity that glider 1 had before collision.

(2) Possibly $v_2 = 0$. This could happen if glider 1 missed glider 2 altogether so that no collision occurred.

(3) Possibly both v_1 and v_2 are both zero. This is not a real possibility because it is known that u_1 was a real velocity at the start of the collision and clearly

$$u_1 \neq 0 + 0$$

Hence, (3) is not to be considered.

$$\boxed{\begin{gathered}\frac{1}{2}\,mu_1^2 = \frac{1}{2}\,mv_1^2 + \frac{1}{2}\,mv_2^2 \\[2ex] u_1^2 = v_1^2 + v_2^2\end{gathered}}$$

FIGURE ⑤

$$\boxed{\begin{gathered}u_1 = v_1 + v_2 \\ u_1^2 = v_1^2 + v_2^2\end{gathered}}$$

FIGURE ⑥

$$\begin{aligned} u_1^2 &= v_1^2 + 2v_1v_2 + v_2^2 && \text{(linear equation squared)} \\ u_1^2 &= v_1^2 + v_2^2 \end{aligned}$$

Subtracting

$$0 = 2v_1\,v_2$$

so $v_1 = 0$ or $v_2 = 0$ or both v_1 and $v_2 = 0$

FIGURE ⑦

(Figure 8) Observe how this simple statement shows that if the velocity of glider 1 after the collision ($v_1 = 0$) is zero, then the velocity of glider 2 after the collision must be the same as the velocity of glider 1 before impact just as was stated above.

(Figure 9) Here again, the substitutions indicate what happens if the velocity of glider 2 after the collision is zero ($v_2 = 0$). It turns out that $u_1 = v_1$ which merely means that glider 1 does not change its velocity at all, having missed impact with glider 2.

As a final step in this discussion, the student is asked to use similar reasoning to determine for himself what would happen if the two gliders became firmly linked together when the collision occurs. Assume that the springs on the gliders are replaced by magnets; glider 2 is at rest and a collision occurs when glider 1 is moving with velocity u_1; the gliders stick to one another and move off after the collision with some velocity $\underline{v}$. It must be remembered that this is an inelastic collision so that kinetic energy is not conserved.

Unless the answer given below results, an error has been made in either concept or mathematics or both.

$$v = 1/2\ u_1$$

1F $v_1 = 0_1$ then

$u_1 = 0 + v_2$

$u_1 = v_2$

FIGURE ⑧

1F $v_2 = 0$, then

$u_1 = v_1 + 0$

$u_1 = v_1$

FIGURE ⑨

COLLISIONS

TERMINAL OBJECTIVES

7/1 A Analyze a two-body collision problem in terms of the impulse mentum theorem.

7/1 C Apply the principle of conservation of momentum to the solution of problems involving inelastic collision.

GRAVITATION

The subject of gravitation and the Cavendish apparatus used to determine the value of G is thoroughly discussed in most college texts. The objective of this paper is a matter of highlighting aspects of the Law of Universal Gravitation which often cause confusion, and enriching the text material by adopting a somewhat different point of view.

(Figure 1) Students often are guilty of paying too little attention to the rigorous implications of the verbal statement of the Law of Universal Gravitation and its mathematical counterpart. The word "object" implies a real body having definite dimensions and mass. How does one measure the distance between such bodies? If the object is perfectly symmetrical, the distance r is measured between geometric centers but when any degree of asymmetry exists, the measurement must be taken between the centers of mass of the respective objects. It should be observed the statement refers to mass, not weight, and the proper units must be employed if numerical results are to be meaningful. To use the law with MKS units, the masses must be expressed in kilograms and the distance of separation in meters; the force of gravitation F will then come out in newtons.

The symbol "G" represents the constant of proportionality and is generally referred to as the "constant of universal gravitation". It must not be confused with "g", the symbol for gravitational acceleration. While "g" is not a constant at all since it varies from place to place even on our own planet, G is a universal constant -- it has the same value regardless of the observer's location in the universe.

"Every object in the universe attracts every other object with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers"

$$F = G\frac{m_1 m_2}{r^2}$$

FIGURE ①

(Figure 2) Students are sometimes puzzled by the fact that G has very specific units of its own, and is certainly not unity while proportionality constants in other equations are dimensionless and are assigned a value of one. For example, in the development of Newton's Second Law, it is first stated that F is directly proportional to the product of m and a using the proportionality symbol as shown. The proportionality symbol is then replaced by an equals sign after inserting the constant of proportionality k. Finally, units are assigned to m and a (in MKS, kg and m/sec^2 respectively, k is allowed to equal unity and be dimensionless, and the resulting force F made to assume the unit obtained from the product of m and a). In MKS units this product unit is kilogram-meters per second per second or $kg \cdot m/sec^2$ which is re-named the newton. Thus the newton is uniquely defined as a $kg \cdot m/sec^2$ and cannot henceforth be defined in any other way.

If a value of 1 kg is substituted for each of the two masses m_1 and m_2 and a distance of 1 meter for r in the gravitational equation

$$F = G \frac{m_1 m_2}{r^2}$$

the force F does not turn out to be 1 newton, neither numerically nor dimensionally. This indicates that the numerical value of G cannot be unity, nor can it be dimensionless. The question then arises as to how one may determine the value of G.

(Figure 3) Can G be mathematically evaluated? When the gravitational equation is solved for G it takes the form shown in the figure. This is of little help mathematically because the force F is still an unknown despite the fact that the masses and the separation may be readily established. Evidently, it is necessary to determine G by experimental methods since the force F must be measured before G can be numerically evaluated. For masses normally encountered in the laboratory, F is extremely minute in magnitude. The apparatus required to measure it, therefore, must be correspondingly sensitive. For example, the gravitational force between two 10-g masses separated by as little as 0.1 meter is less than one-billionth of a newton!

(a) $F \propto ma$

(b) $F = kma$

(c) $F = ma$ since $k = 1$ if m is in kilograms, a is in m/sec and F is in newtons.

FIGURE ②

$$\boxed{G = \frac{m_1 m_2}{Fr^2}}$$

FIGURE ③

(Figure 4) This is a schematic diagram of the apparatus used by Henry Cavendish in 1798 to measure the value of G. It is a torsional balance of great sensitivity. Two spherical masses m_1 and m_2 at the ends of the cross-bar of a rigid T-frame are free to move when a force is applied to either or both if this force has a component at right angles to the cross-bar. Mounted on the vertical leg of the frame is a light mirror, the assembly being supported in space by a fine quartz thread or a metallic ribbon. Two massive spheres, usually of lead, are placed near the masses at the ends of the cross-bar (m_1 and m_2). The entire system is then given time to stabilize and come to complete rest. At this point, the light source is adjusted so that its beam is reflected from the mirror to the scale; the scale reading is recorded. It should be clear that this assembly constitutes an optical lever which magnifies even a tiny deflection of the mirror so that it is readily measurable on the scale. The two large masses (m_2) are now in position one.

Very, very carefully the m_2 masses are then moved into their respective second positions. This reverses the torque applied to the cross-bar since the gravitational force due to the attraction of each m_1 and its corresponding m_2 has been reversed in direction. The cross-bar begins to twist on the suspension and, as might be expected, overshoots its ultimate final position so that it goes into damped oscillation like a torsional pendulum. The time required for the system to stabilize once more may be as long as two hours in a typical laboratory set-up. Once it has again come to complete rest, the angle of twist is easily measured by observing the new position of the light spot and utilizing the geometry of the system.

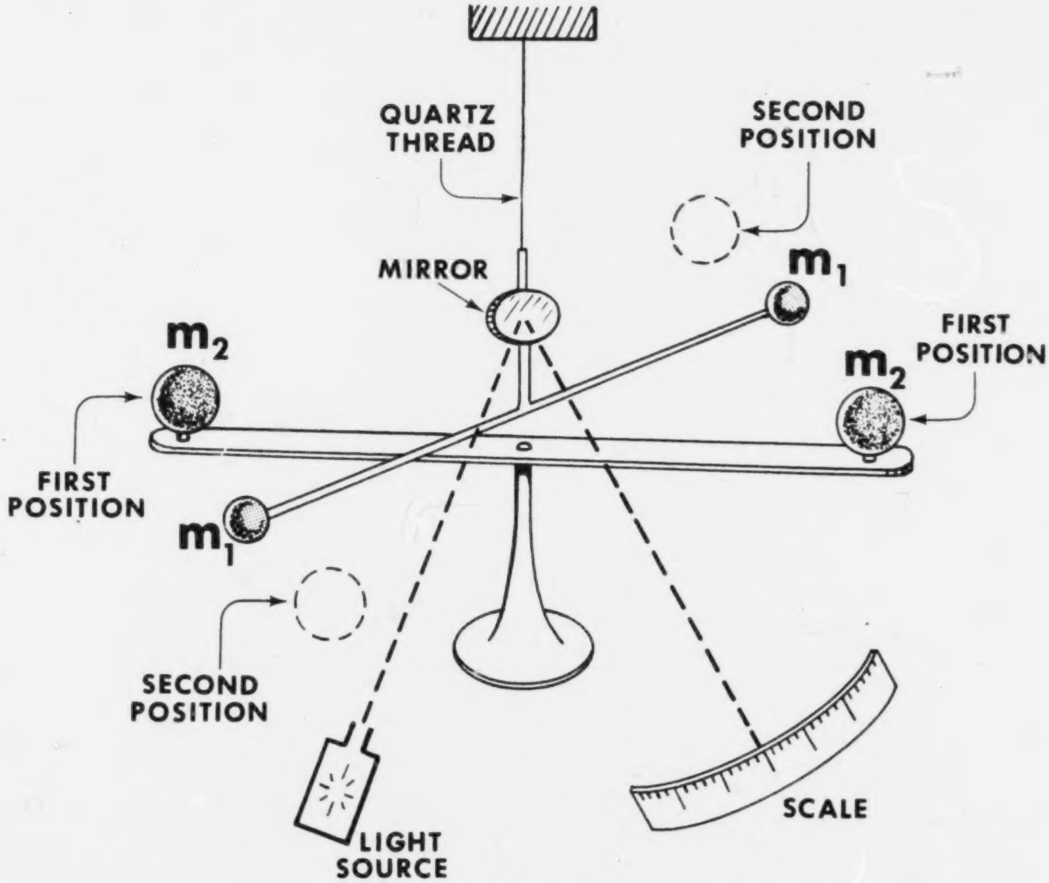

QUARTZ THREAD
SECOND POSITION
MIRROR
m_1
m_2
FIRST POSITION
m_2
FIRST POSITION
m_1
SECOND POSITION
LIGHT SOURCE
SCALE

FIGURE ④

(Figure 5) The relationship between G and the other relevant quantities is shown in this equation. Its derivation is not particularly difficult but it is based on information that the student has not yet had so that it will be passed over at this point. The symbol "k" represents the torsional constant of the thread or ribbon and is experimentally found by methods that do not involve gravity or gravitational forces. The symbol "L" stands for the length of the cross-bar measured between centers of gravity of m_1 masses. With all of the quantities now known or measurable, the numerical value of G may now be obtained.

(Figure 6) Clearly G is not dimensionless as this development indicates. The student should check the substitutions shown carefully. The unit for k is the $kg \cdot m^2/sec^2$; for θ it is the radian (dimensionless); for r^2 it is the $meter^2$; for m_1 and m_2 it is the kg; and for L it is the meter. The student is also asked to show why these substitutions result in a final unit for G equal to the $nt \cdot m^2/kg^2$.

As a final suggestion, the student is asked to substitute this unit for G into the equation

$$F = G \frac{m_1 m_2}{r^2}$$

to show that the force of gravitation F does turn out to be measurable in newtons.

$$G = \frac{k \theta r^2}{m_1 m_2 L}$$

where: k = torsional constant of suspension thread

θ = angle of twist

r = distance from center of m_1 to center of m_2

L = length of horizontal bar

FIGURE ⑤

$$G = \frac{k \theta r^2}{m_1 m_2 L}$$

$$G = \frac{\frac{kg \cdot m^2}{sec^2} \cdot radians \cdot m^2}{kg \cdot kg \cdot m}$$

$$G = \frac{nt \cdot m^2}{kg^2}$$

FIGURE ⑥

GRAVITATION

TERMINAL OBJECTIVES

8/1 A Analyze gravitational force actions between two particles in terms of the gravitational field.

I

CALCULATION OF $\vec{E}$ FOR AN INFINITE UNIFORMLY CHARGED WIRE

At this point in his studies, the reader should be cognizant of the importance of being able to determine the electric field due to any distribution of charge. It is not unusual for this to be a fairly difficult problem. To the contrary, however, if the charge distribution has a high degree of symmetry, the problem may be substantially simplified.

In the following material, an idealized case with a very high degree of symmetry will be presented; that is, an infinite wire with its charge uniformly distributed over its length. This is the same as saying that the charge per unit length is constant over the wire.

The reader should keep in mind that there are two salient points to his study of this problem. Firstly, the result to be obtained has its own intrinsic importance. Secondly, but of equal importance is the fact that this problem will give the reader an excellent example of the applications of the integral calculus to the solution of practical problems of physics.

In general terms, the procedure in solving this problem will be to determine the electric field contribution from an infinitesmal element of charge. Upon doing that, a summation will be taken over all the elements of charge. This summation will require the use of the integral calculus.

First, a general over view of the organization of the problem will be given. Following that, the solution will be shown in considerable detail.

In Figure 1, an infinite wire is represented by the vertical line. The solution of the problem will involve calculating the electric field $\vec{E}$ at point P due to the charge on the wire. It is assumed that the charge on the wire is positive. The line $\underline{a}$ represents the perpendicular distance from the point P to the wire. Vertical distances along the wire will be measured by the variable $\underline{y}$. The origin of measurements along $\underline{y}$ will be the foot of point P, that is the point at which the line $\underline{a}$ forms a right angle with the wire.

Now, if along the distance $\underline{y}$, there is an element of length dy, this element will carry a charge. Since the charge is linear over the wire, a linear charge density λ is defined. λ will then be equal to the charge per unit length of wire. Hence, the total charge on a section of wire will be given by the product of the linear charge density λ and the length of wire being considered. Thus, the charge along the element of length dy is λ dy.

Going back to Figure 1, note that $\underline{r}$ represents the distance from the element of length dy to the point P. Also shown in Figure 1 is the angle Θ, (which is the angle between $\underline{a}$ and $\underline{r}$), and $\underline{d}$ Θ which is the angle substended by the element of length at the point P. These are the important variables and constants in the problem.

At this point, the reader should study the presentation above very carefully!

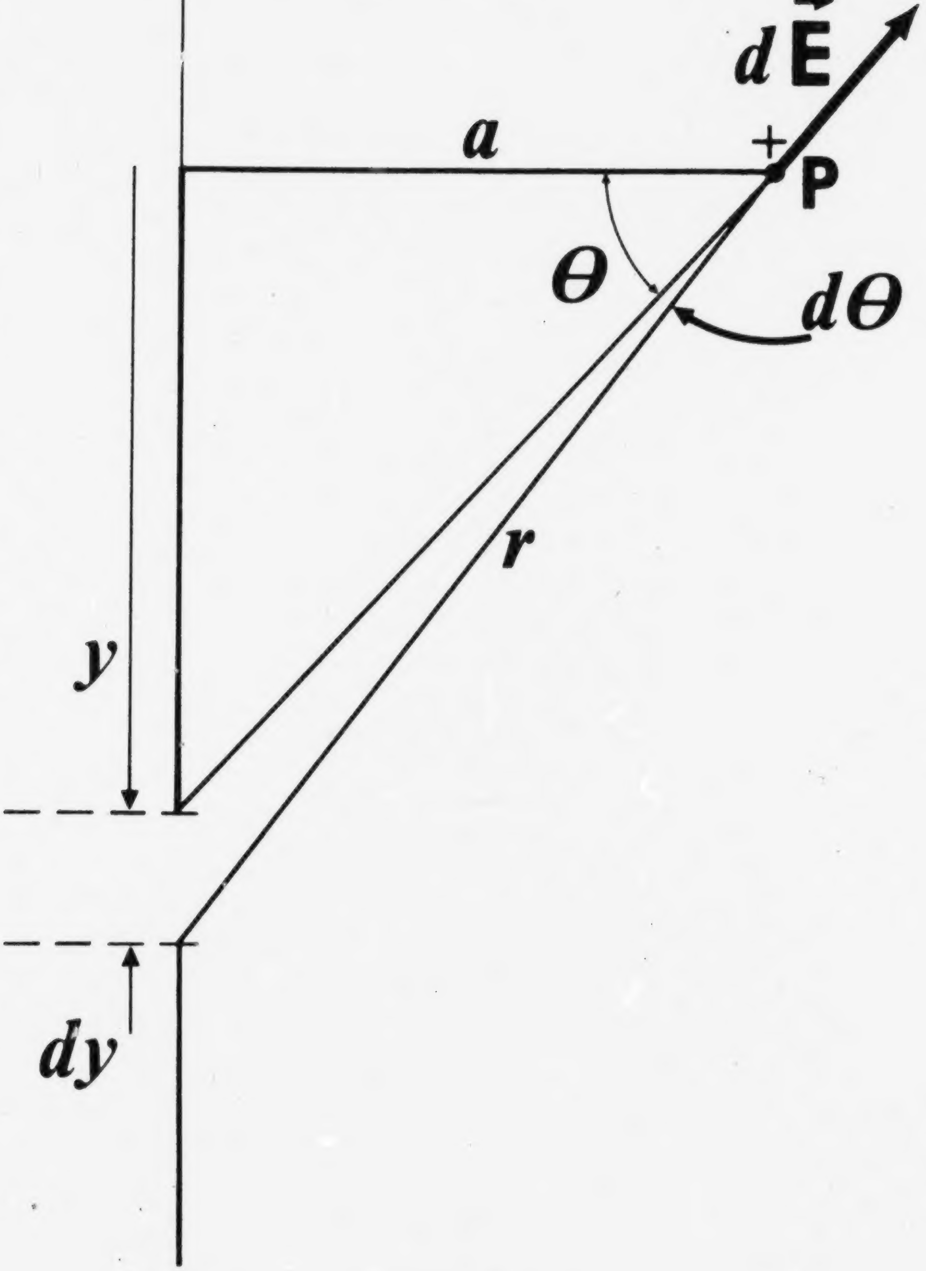

$d\vec{E}$
a
+
P
θ
$d\theta$
r
y
dy

FIGURE ①

The following discussion revolves around Figure 2.

For a point charge, $\underline{d}\,\vec{E}$ at a point is given by

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} \quad \frac{\lambda dy}{r^2} \quad \hat{r}$$

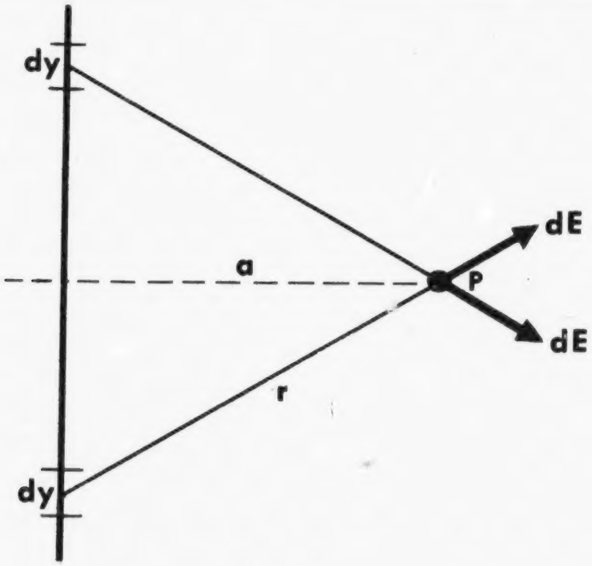
where $\underline{\hat{r}}$ is a unit vector from dy in the direction of P.

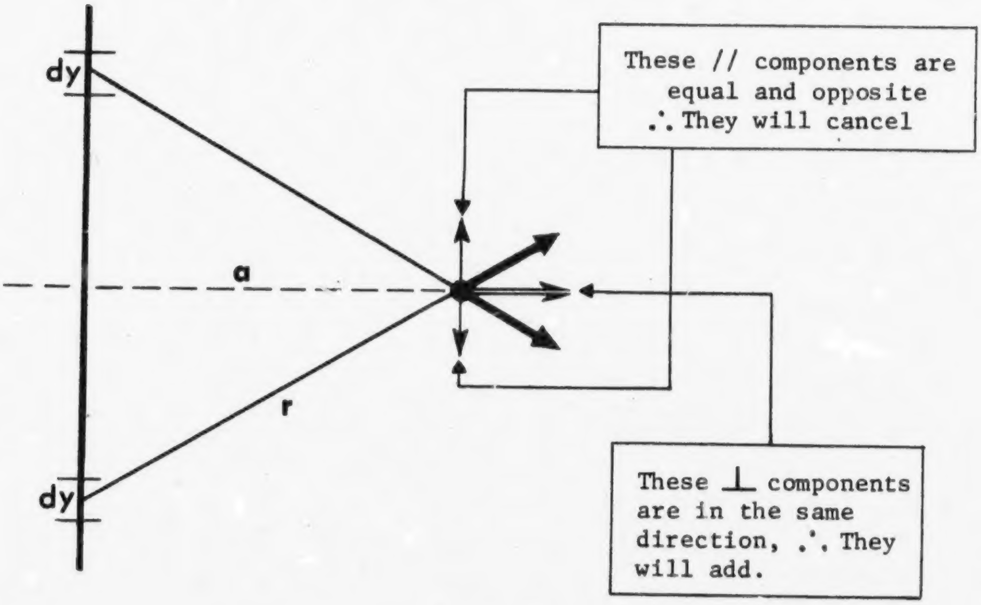
The next step is to consider the symmetry of the problem. The following discussion will center about Figure 3.

Consider elements of length dy that are both above and below the foot of perpendicular $\underline{a}$. Each of these elements of length will contribute to the electric field at point P. If one takes components of $\underline{d}\,\vec{E}$ both parallel to and perpendicular to the wire, one sees that the components of $\vec{E}$ parallel to the wire are of equal magnitude but are oppositely directed; thus, these parallel components will cancel, and one will be left with the perpendicular components only. The perpendicular components will add. The obvious conclusion is that, since the entire wire may be considered to be made up of such pairs of elements, the electric field at point P must be in the x-direction. The x- direction in this analysis is defined as being parallel to line $\underline{a}$, or perpendicular to the length of the wire. Henceforth, consideration need be given only to the x- component of $d\vec{E}$, namely dE_x.

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\,\frac{\lambda\, dy}{r^2}\,\hat{r}$$

FIGURE ②





These // components are equal and opposite ∴ They will cancel

These ⊥ components are in the same direction, ∴ They will add.

FIGURE ③

A study of Figure 4 clearly shows that

$$(1) \qquad dE_x = dE \cos \Theta$$

The equation for dE at a point is repeated in equation (2)

$$(2) \qquad d\vec{E} = \frac{1}{4\Pi\varepsilon_o} \frac{\lambda dy}{r^2} \hat{r}$$

Upon substituting equation (2) into equation (1), one obtains

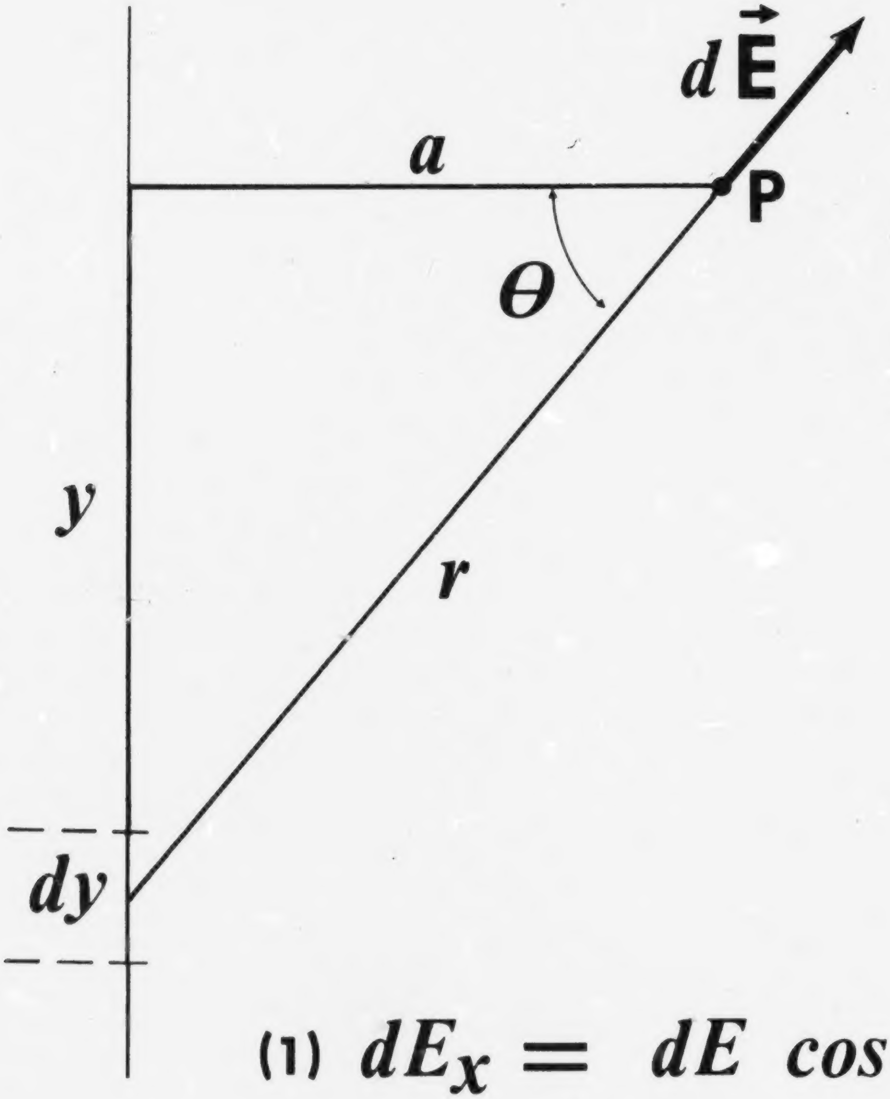
$$(3) \qquad dE_x = \frac{1}{4\Pi\varepsilon_o} \frac{\lambda dy}{r^2} \cos \Theta$$

Note that since this is a scalar equation, the vector notation has been omitted.

Note that the angle Θ in Figure 4 will be taken as negative. Angles clockwise from line a will be taken in the positive sense. Note also that Figure 4 makes it clear that

$$\cos \Theta = \frac{a}{r}$$

This relation will be important later, since equation (3) involves three variables: y, r, and Θ. In such a form, equation (3) is not readily integrable. In order for the integration to proceed, two of the three variables will have to be expressed in terms of the third. The above cosine relation will allow this to be done.


$d\vec{E}$
a
P
θ
y
r
dy

(1) $dE_x = dE \cos\theta$

(2) $d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dy}{r^2} \hat{r}$

(3) $dE_x = \frac{1}{4\pi\epsilon_0} \frac{\lambda dy}{r^2} \cos\theta$

FIGURE ④

The method of making these substitutions is shown in Figure 5 using the elemental triangle made by dy subtending an angle dθ at the point P. The element dl is shown as a perpendicular dropped from one radius vector to the other. Note that dl, dy, and the undesignated segment of r form a small right triangle. From the above cosine relation, r may be written as

$$r = \frac{a}{\cos\theta}$$

From a study of the elemental triangle, it can be seen that

$$dl = r\,d\theta$$

and

$$\frac{dl}{dy} = \cos\theta$$

Thus

$$dy = \frac{r\,d\theta}{\cos\theta}$$

which is an expression for dl and dy in terms of r and θ. Thus in the final expression, r may be eliminated to yield an expression for dy in terms of θ and a (a constant).

Continuing with the substitution, the identity above may replace dy into equation (3) yields

$$dE_x = \frac{1}{4\Pi\varepsilon_o}\;\frac{\lambda\cos\theta}{r^2}\;\frac{r\,d\theta}{\cos\theta}$$

$$= \frac{1}{4\Pi\varepsilon_o}\;\frac{\lambda\,d\theta}{r^2}$$

But, recall that

$$r = \frac{a}{\cos\theta}$$

which gives for dE_x

$$dE_x = \frac{1}{4\Pi\varepsilon_o}\;\frac{\lambda\cos\theta}{a}\;d\theta$$

Expressing this differential equation in integral form, one obtains

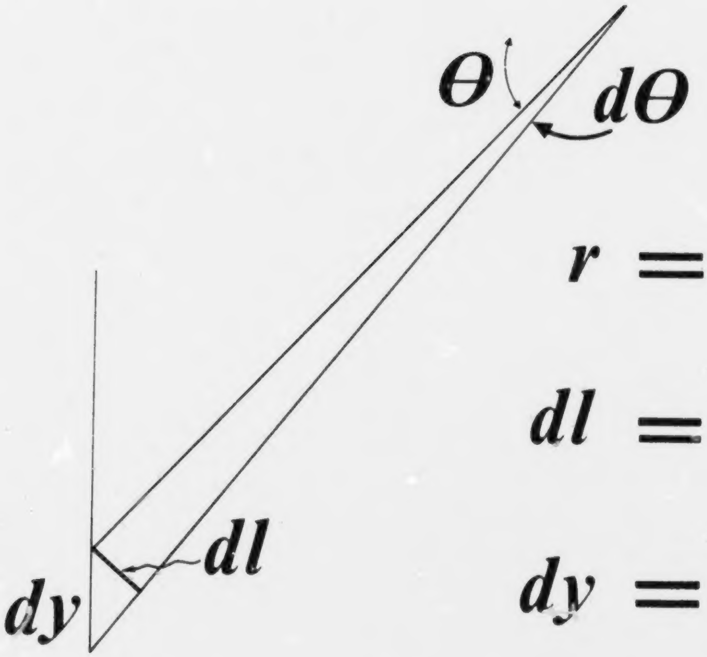
$$E_x = \frac{1}{4\Pi\varepsilon_o}\;\frac{\lambda}{a}\int_{\theta_1}^{\theta_2}\cos\theta\;d\theta$$

Note that since λ and a are constants, they appear outside the integral sign. The integral is easily evaluated since the integral of cos θ is sin θ. Upon performing this operation on the above equation for E_x, one obtains

$$(4)\qquad E_x = \frac{1}{4\Pi\varepsilon_o}\;\frac{\lambda}{a}\;(\sin\theta_2 - \sin\theta_1)$$

Since θ_2 is 90°, $\sin\theta_2 = 1$; and since θ_1 is − 90°, $\sin\theta_1 = -1$, then equation (4) becomes

$$E_x = \frac{1}{2\Pi\varepsilon_o}\;\frac{\lambda}{a}$$


θ
$d\theta$
dl
dy

$$r = \frac{a}{\cos\theta}$$

$$dl = r\, d\theta$$

$$dy = \frac{r\, d\theta}{\cos\theta}$$

FIGURE ⑤

Recall that λ is the charge per unit length, and as such is proportional to the total charge on the wire. Thus E_x is proportional to the total charge on the wire. Recall also that a is the perpendicular distance from the point at which the field due to the wire is being determined.

From this information some important conclusions may be stated with regard to the field generated by this type of charge distribution. The field is inversely proportional to the first power of the distance. This is different from the expressions that the reader has met before which have all involved inverse square laws. It is important to note that this is not an inverse square law but a simple inverse law.

The mathematical features of the derivation of the field due to an infinite uniformly charged wire are summarized in Figure 6.

$$dE_x = \frac{1}{4\pi\epsilon_o} \frac{\lambda \cancel{\cos\theta}}{r^{\cancel{2}}} \frac{\cancel{r}\, d\theta}{\cancel{\cos\theta}}$$

$$= \frac{1}{4\pi\epsilon_o} \frac{\lambda \cos\theta\, d\theta}{a}$$

$$E_x = \frac{1}{4\pi\epsilon_o} \frac{\lambda}{a} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta$$

$$= \frac{1}{4\pi\epsilon_o} \frac{\lambda}{a} \left[\sin\theta_2 - \sin\theta_1 \right]$$

For a infinitely long wire: $\theta_2 = 90^\circ$
$\theta_1 = -90^\circ$

$$E_x = \frac{1}{2\pi\epsilon_o} \frac{\lambda}{a}$$

FIGURE ⑥

CALCULATION OF $\vec{E}$ FOR AN INFINITE UNIFORMLY CHARGED WIRE

TERMINAL OBJECTIVES

10/2 B Answer questions and solve problems relating to atomic models based on sperically symmetric charge distributions.

11/1 A Solve problems and answer questions on the relationship between potential and field intensity.

I

DEFLECTION OF ELECTRONS IN AN ELECTRIC FIELD

Some of the most important technological advances of recent years have stemmed directly from our knowledge and understanding of the way in which electrons in motion are affected by passing through an electric field. This paper will be concerned with a thorough analysis of the forces acting on a parallel beam of electrons moving through a uniform electric field. The discussion is based on the observations that can be made of the motion of the fluorescent spot seen on the screen of a cathode ray tube.

(Figure 1) The cathode ray tube illustrated in this drawing is a demonstation type in which electrons are emitted thermionically from the heated cathode. Those electrons which pass into the focusing cylinder are formed into a parallel beam which is collimated into a thin pencil as it moves through the aperture in the anode. The beam is then injected into the space between the electrodes labeled "deflection plates" and proceeds onward to the fluorescent screen where it produces a visible spot of light. The dimensions given for the length 1 of each of the plates, the distance L from the edge of either plate to the screen, and the spacing between the deflection plates are representative values that correspond to the actual dimensions of the elements of the tube shown in the diagram.

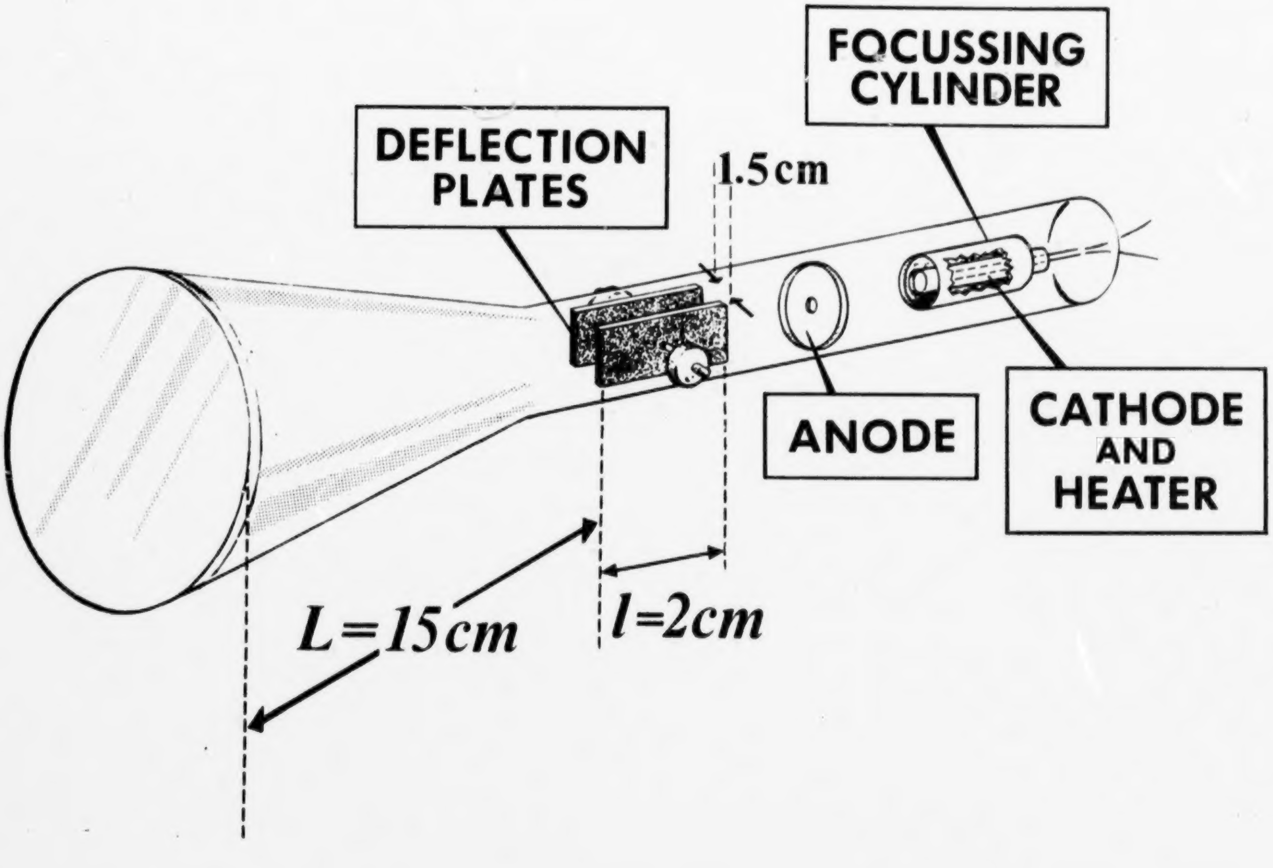

FOCUSSING CYLINDER
DEFLECTION PLATES
1.5cm
ANODE
CATHODE AND HEATER
L=15cm
l=2cm

FIGURE ①

(Figure 2 This is a close-up of the electron "gun" of the tube showing the normal potential difference used between cathode and anode. The anode is 250 volts positive with respect to the cathode, hence electrons emitted from the cathode are greatly accelerated in the direction of the anode. Once the electron beam has passed the anode, its motion horizontally along the axis of the tube becomes uniform since it is no longer in the space where the potential gradient exists in this direction.

(Figure 3) Consider the electrons just as they are emitted from the cathode. At this point they have a specific amount of potential energy due to the voltage gradient. When they arrive at the anode, all of this potential energy has been converted to kinetic energy. Thus, as the beam passes through the aperture, the amount of kinetic energy gained is equal to the amount of potential energy lost in transit.

(Figure 4) Since the potential difference V between cathode and anode is actually potential energy per unit charge, the magnitude of the potential energy of the electron is thus given as eV in which e is the charge on the electron. As shown above, this must equal the kinetic energy at the anode as indicated by the upper equation. Note that v_h symbolizes the "horizontal" component of the beam velocity at the anode, or the component parallel to the axis of the tube. When solved for v_h, the result indicated in the second equation is obtained. In this equation, m is the mass of the individual electron. All the quantities on the right are readily measurable so that v_h may be easily evaluated.

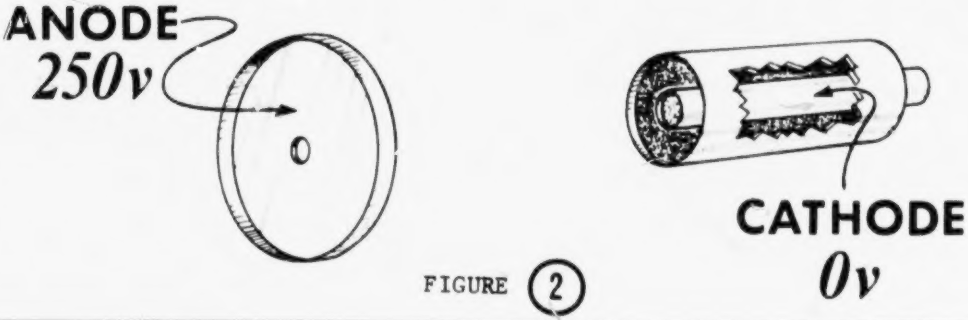
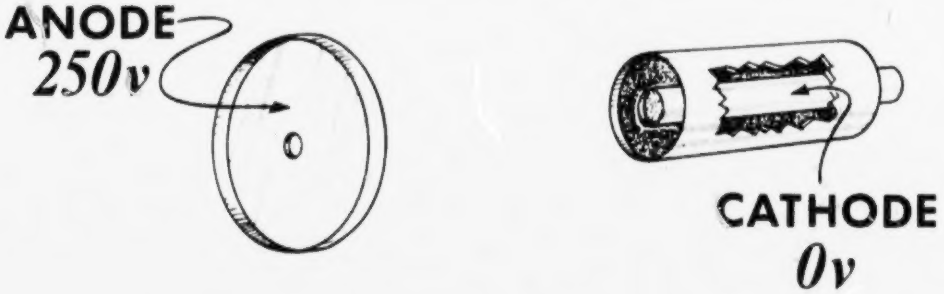
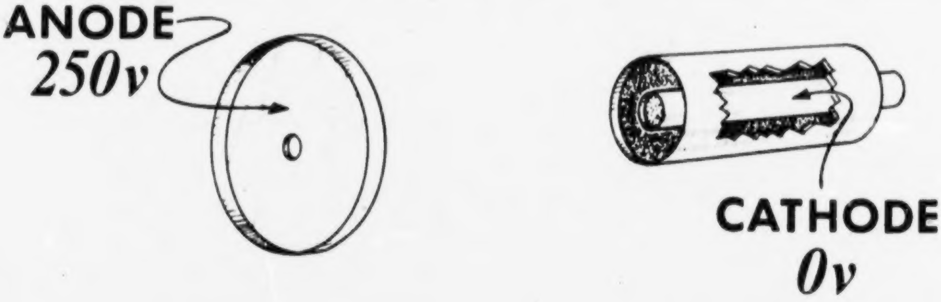

ANODE
250v
CATHODE
0v

FIGURE ②


ANODE
250v
CATHODE
0v

Loss in Potential Energy* = *Gain in Kinetic Energy

FIGURE ③


ANODE
250v
CATHODE
0v

Loss in Potential Energy* = *Gain in Kinetic Energy

$$eV = \frac{1}{2} m v_h^2$$

$$v_h = \sqrt{\frac{2eV}{m}}$$

FIGURE ④

(Figure 5) This diagram suggests the next step in the analysis. The electron beam approaches the deflection plates traveling along the axis of the cathode ray tube. If no potential difference is established between the plates, the beam will proceed through the space between them with zero deviation and produce a light spot at the exact center of the screen. Should a potential difference be applied to the plates, the resulting electric field between them would then have to be taken into account in determining the effect on the beam path.

(Figure 6) In this plan view of the deflection plates, assume that the upper plate has been negatively charged with respect to the lower plate, establishing an electric field having the direction shown, that is, from the positive toward the negative plate. Recalling that electrons are negatively charged particles, the beam would experience a force opposite that of the direction of the field. In this view, the force on the beam would be downward, toward the positive plate.

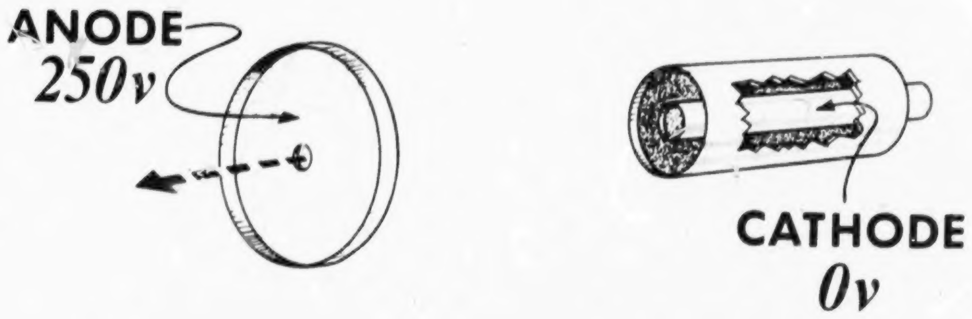

ANODE
250v
CATHODE
0v

FIGURE ⑤

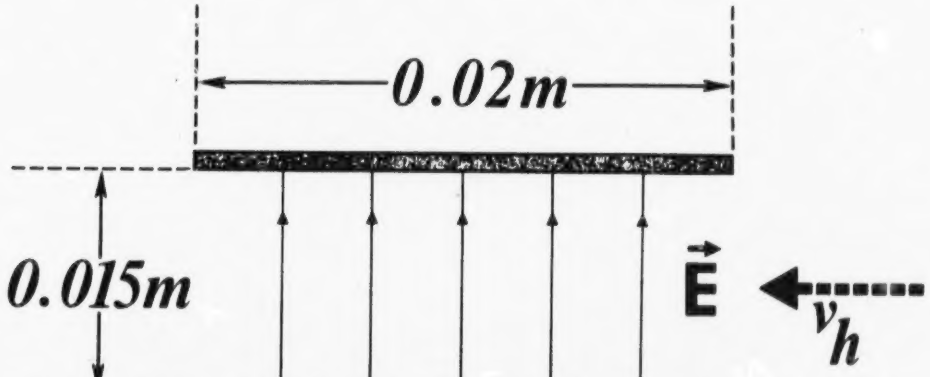

0.02m
0.015m
$\vec{E}$
v_h

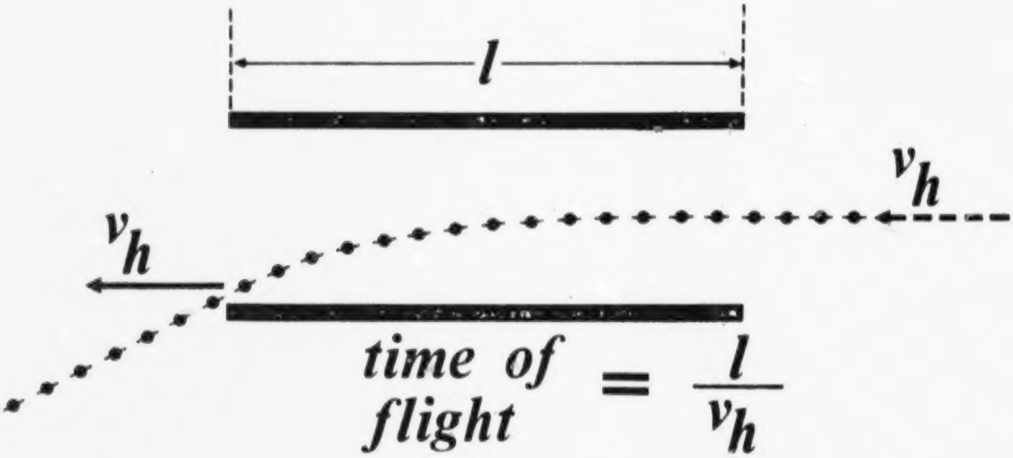
FIGURE ⑥

(Figure 7) As a result of this force, the beam would be deflected downward and follow a curved trajectory while in the space where the potential gradient exists. It is important to observe that the beam path is a straight line in the range from the cathode to the right edge of the deflection plate, a parabola in the deflection area, and a straight line once again after the beam has passed the left edge of the plate. Note also that v_h represents the axial component of the beam velocity at all times after the latter has passed the anode.

(Figure 8) Electric intensity E is defined as force per unit charge. To determine the force on the electron beam due to the electric field, it is merely necessary to multiply force per unit charge by the charge on the electron, or Ee. This makes it possible to express the acceleration of the beam at right angles to the tube axis in terms of electric intensity E, electronic charge e, and the mass of the electron m.

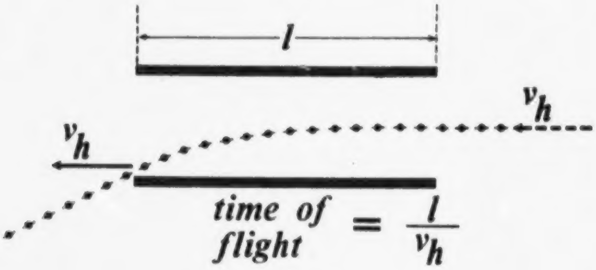
(Figure 9) As indicated here, the transaxial acceleration is given by Ee/m. With this relationship in hand, the transaxial electron beam displacement may now be evaluated by substituting in the general equation

$$\text{displacement} = 1/2\ a\ t^2$$



$$\text{time of flight} = \frac{l}{v_h}$$

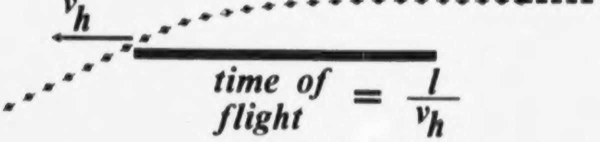
FIGURE ⑦



$$\text{time of flight} = \frac{l}{v_h}$$

$$\text{force on electrons} = Ee$$

FIGURE ⑧



$$\text{time of flight} = \frac{l}{v_h}$$

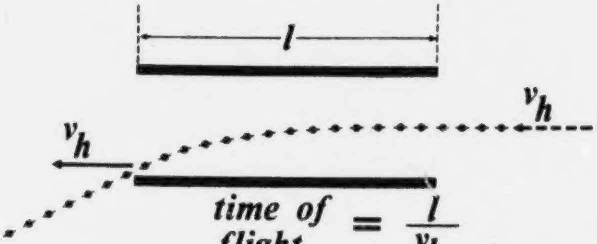
$$\text{force on electrons} = Ee$$

$$\text{acceleration} = \frac{F}{m} = \frac{Ee}{m}$$

FIGURE ⑨

(Figure 10) The added equation presents this information. The term Ee/m replaces a in the general statement, and $(\ell/v_h)^2$ represents the replacement for t^2. The time in this equation is the distance traveled along the axis (ℓ) divided by the axial velocity of the beam (v_h). Before proceeding further, it will be necessary to determine the transaxial velocity of the electron just as it reaches the left edge of the deflection plate.

(Figure 11) The transaxial velocity at the point indicated is obtained from the general relationship $v = at$ which applies to any body having a uniform acceleration a for a time t. The deflection acceleration is Ee/m as indicated once again in this figure. The time of flight in the deflection region is ℓ/v_h .



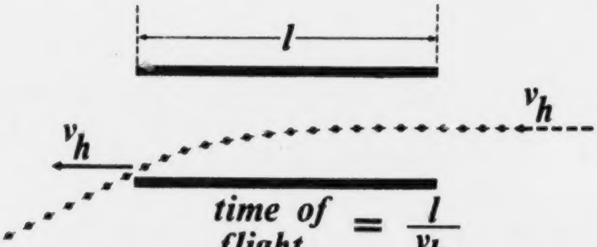
$$\text{time of flight} = \frac{l}{v_h}$$

$$\text{force on electrons} = Ee$$

$$\text{acceleration} = \frac{F}{m} = \frac{Ee}{m}$$

$$\textbf{deflection in plate region} = \frac{1}{2}at^2 = \frac{1}{2}\frac{Ee}{m}\left(\frac{l}{v_h}\right)^2$$

FIGURE ⑩



$$\text{time of flight} = \frac{l}{v_h}$$

$$\text{force on electrons} = Ee$$

$$\text{acceleration} = \frac{F}{m} = \frac{Ee}{m}$$

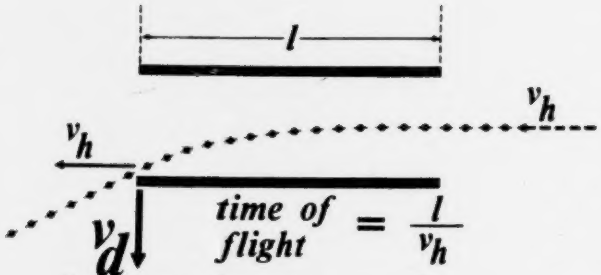
$$\text{deflection in plate region} = \frac{1}{2}at^2 = \frac{1}{2}\frac{Ee}{m}\left(\frac{l}{v_h}\right)^2$$

$$\textbf{deflection acceleration} = \frac{Ee}{m}$$

FIGURE ⑪

(Figure 12) Therefore, the transaxial velocity at the left edge of the plate is given by this additional relationship. Essentially, this implies that the electron is moving with an axial velocity v_h (uniform) when it reaches the left plate edge, and with a transaxial velocity v_d at the same instant. To compute the displacement of the fluorescent spot from its central position as a result of the deviation, it is now necessary to determine the additional transaxial displacement of the beam as it travels from the left edge of the plate to the screen. It should be recalled that there is no transaxial force on the beam in this region; its trajectory is a straight line.

(Figure 13) This additional deflection is the product of the drift time (time form motion from plate to screen) and the transaxial velocity v_d (now uniform). The drift time is simply L/v_h so that the additional deflection is merely $L/v_h \times v_d$. Thus, information is now available for finding the deflection of the beam while in the plate region and also the deflection between the plate region and the screen. The total deflection is, of course, the sum of these.



$$\text{time of flight} = \frac{l}{v_h}$$

$$\text{force on electrons} = Ee$$

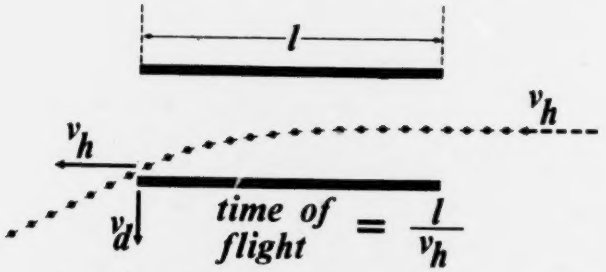
$$\text{acceleration} = \frac{F}{m} = \frac{Ee}{m}$$

$$\text{deflection in plate region} = \frac{1}{2}at^2 = \frac{1}{2}\,\frac{Ee}{m}\left(\frac{l}{v_h}\right)^2$$

$$\text{deflection acceleration} = \frac{Ee}{m}$$

$$\therefore \text{final deflected velocity} = \frac{l}{v_h}\,\frac{Ee}{m}$$

FIGURE ⑫



$$\text{time of flight} = \frac{l}{v_h}$$

$$\text{force on electrons} = Ee$$

$$\text{acceleration} = \frac{F}{m} = \frac{Ee}{m}$$

$$\text{deflection in plate region} = \frac{1}{2}at^2 = \frac{1}{2}\,\frac{Ee}{m}\left(\frac{l}{v_h}\right)^2$$

$$\text{deflection acceleration} = \frac{Ee}{m}$$

$$\therefore \text{final deflected velocity} = \frac{l}{v_h}\,\frac{Ee}{m}$$

$$\text{additional deflection} = \text{drift time} \times v_d$$

$$= \frac{L}{v_h}\,v_d$$

FIGURE ⑬

(Figure 14) This figure shows the algebraic solution. The total displacement of the light spot on the screen from its central, undeviated position may be found by substituting in the final equation shown. For the dimensions and electrical values given, the displacement of the spot turns out to be 4.2×10^{-2} meter or 4.2 cm. The measured value obtained in the actual experiment was 4.3 cm, indicating excellent agreement with the calculations.

$$\begin{aligned}
\textit{deflection in plate region} &= \frac{1}{2}\,at^2 = \frac{1}{2}\,\frac{Ee}{m}\left(\frac{l}{v_h}\right)^2 \\
\textit{additional deflection} &= \textit{drift time} \times v_d \\
&= \frac{L}{v_h}\,v_d \\
\textit{total deflection} &= \textit{sum of these} = \frac{Ee}{m}\,\frac{l}{v_h^2}\left(\frac{l}{2}+L\right) \\
&= 4.2\times 10^{-2}\,m
\end{aligned}$$

FIGURE ⑭

DEFLECTION OF ELECTRONS IN AN ELECTRIC FIELD

TERMINAL OBJECTIVES

10/3 B Answer questions and solve problems relating to potential and field strength.

FLUX

In general usage, flux means a "flowing" or a "flow". It is normally applied to a fluid to describe its rate of motion or direction of movement. In physics, "flux" is frequently applied to certain aspects of vector fields but retains the implication of the flow of something. The intimate relationship of the concept of flow with that of flux makes it logical to begin a discussion of this subject with a fluid analogy.

(Figure 1) In this representation of a river, it is assumed that the water is flowing in the general direction of the observer and has attained a steady state with respect to velocity. This means that the water flowing past any given point in the stream has the same velocity second after second. The river may then be visualized as a velocity field because every point in it may be represented by a velocity vector.

(Figure 2) If certain velocities are selected arbitrarily at various levels, their comparative magnitudes may be represented by suitable vector arrows. For simplicity, it has been assumed that the volocity of the water near the surface is less than it is at greater depths, hence the vectors at the top are shorter and are identified with lower case "$\vec{v}$'s" while those at the bottom are longer and are symbolized with upper case "$\vec{V}$'s".

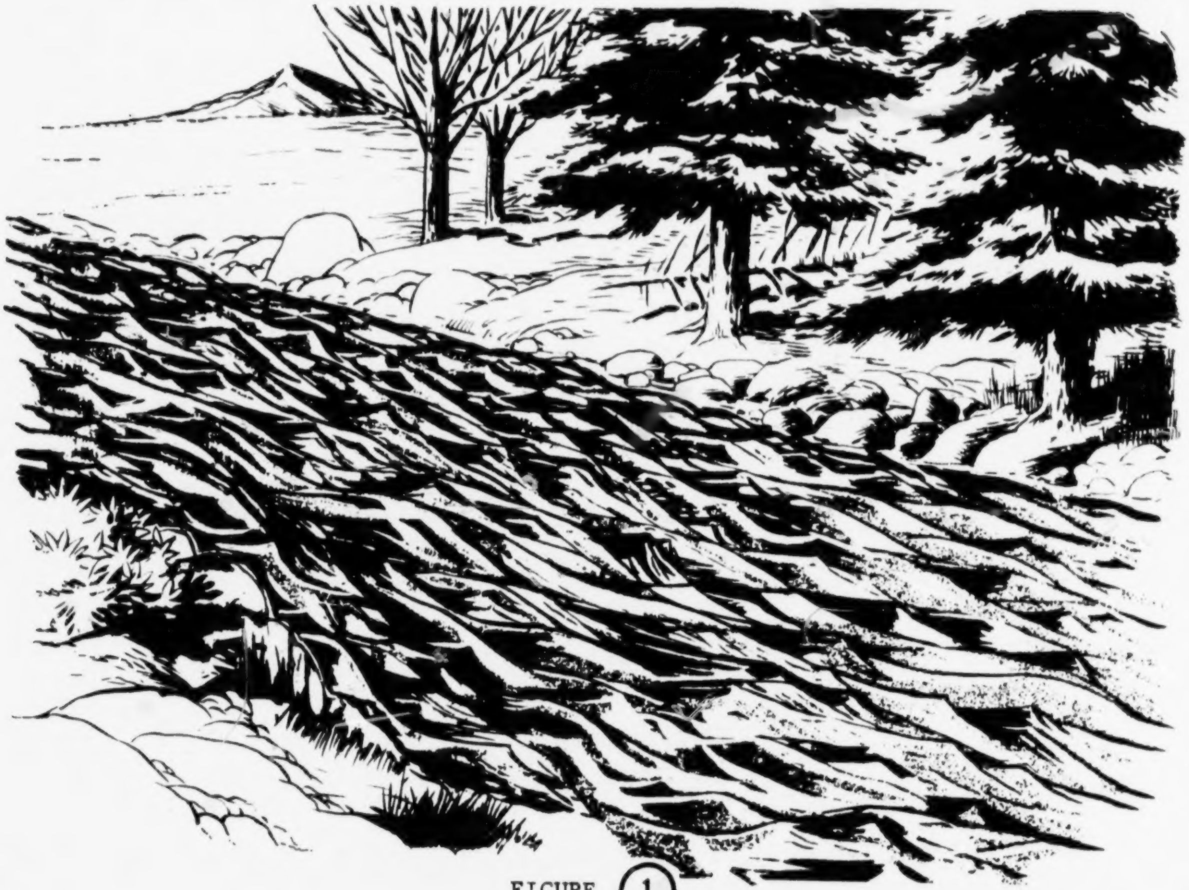


FIGURE ①

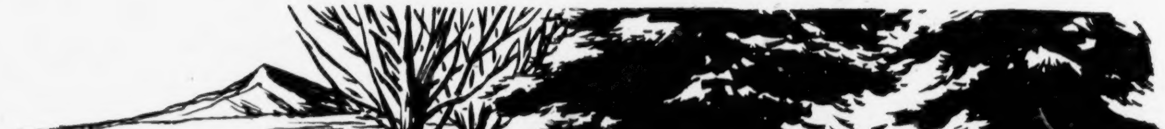


FIGURE ②

(Figure 3) A small area A has been sketched into midstream and has been placed in the low velocity stratum of the river. This area is to be considered extremely small although it is illustrated as enlarged for clarity. In accordance with established convention, the vector that describes this area is drawn perpendicular to it.

(Figure 4) The flux through this area is, by definition, the dot product of the velocity $\vec{v}$ and the area $\vec{A}$. Since this is (by definition!) a dot product, flux is a scalar quantity. The equation for flux may also be written in terms of the components of the vector, or

$$\phi = vA\cos\theta$$

where θ is the angle between the actual velocity vector and the area. It is revealing to analyze the expression for flux dimensionally. Velocity is length per unit time and area is expressed in length units squared. The product of v and A is then

$$\frac{(L)}{(T)} \cdot (L)^2 = \frac{(L)^3}{(T)}$$

Thus, flux has the dimensions of volume per unit time and represents the volume of liquid flowing through the area A per unit time.

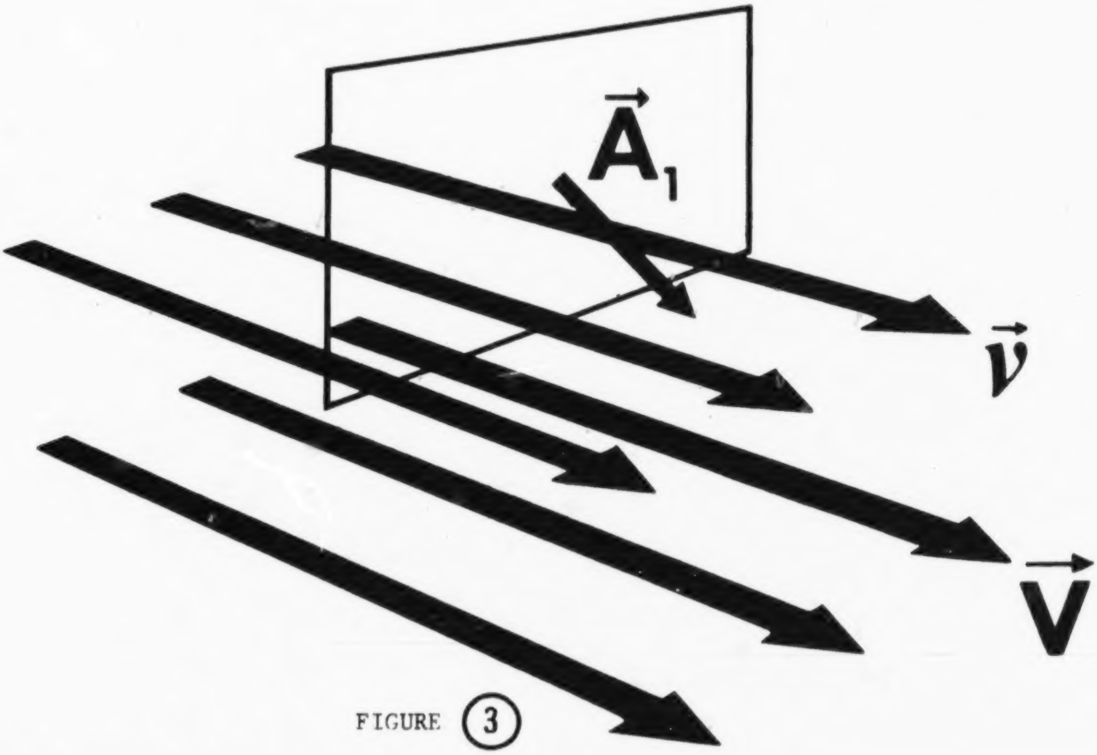

$\vec{A}_1$
$\vec{v}$
$\vec{V}$

FIGURE ③

$$\phi = \vec{v} \cdot \vec{A}$$

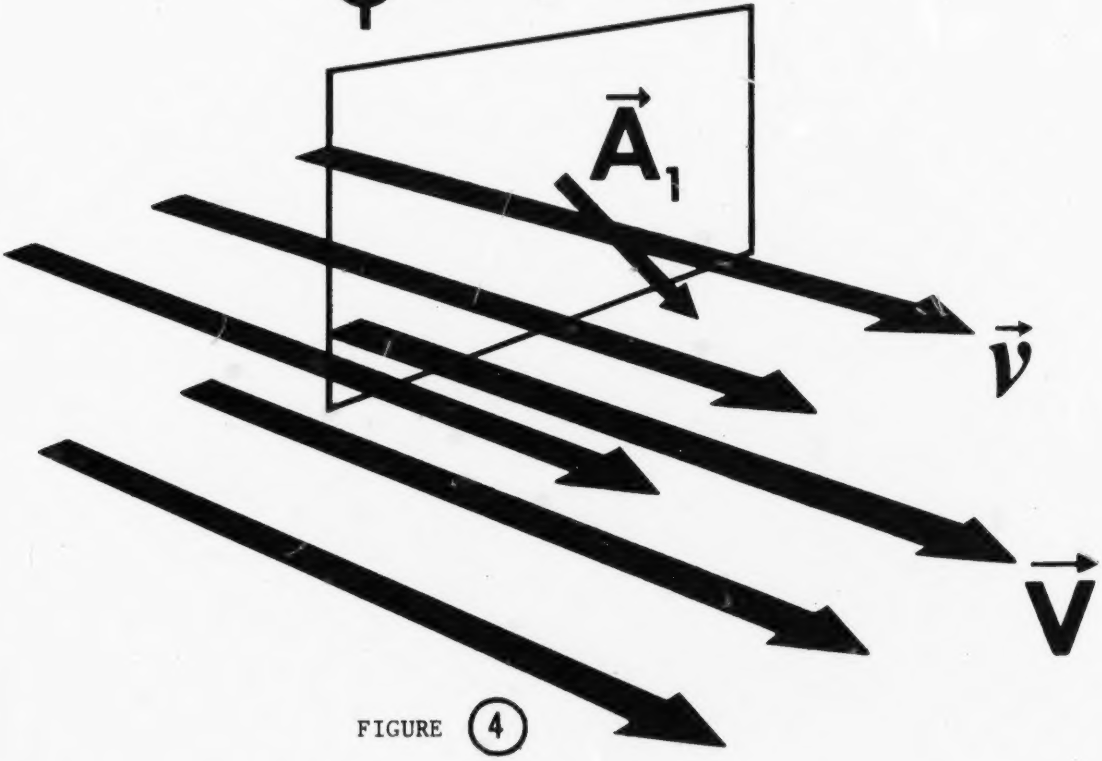

$\vec{A}_1$
$\vec{v}$
$\vec{V}$

FIGURE ④

(Figure 5) This diagram should help to clarify the significance of the angle θ. The area through which the flow occurs is again displayed with it descriptive vector $\vec{A}$ at right angles to the surface; the velocity vector $\vec{v}$ is shown as a horizontal arrow. If the velocity vector were perpendicular to the area vector, there would be zero flux since none of the liquid would be passing through the area; that is, the flux is maximum when the velocity vector and the area vector are parallel to one another. For this condition, the angle θ is zero so that the cosine of the angle would be unity; as the angle becomes larger, the magnitude of the flux diminishes.

(Figure 6) At this point, the area is increased by adding a second surface $\vec{A}_2$ so that the total surface is now the sum of the original area and the newly added portion. It will be assumed that the velocity through the newly added section is vector $\vec{V}$. The total flow through the enlarged area is now equal to the sum of the flows through the separate surfaces.

(Figure 7) This sum relationship has been added to the illustration. The original area is designated as $\vec{A}_1$ and the added surface as $\vec{A}_2$. Although the foregoing development many appear trivial, it does lead up to an important idea: calculation of the flux through a surface involves the process of summation; in the limit, this process becomes one of integration.

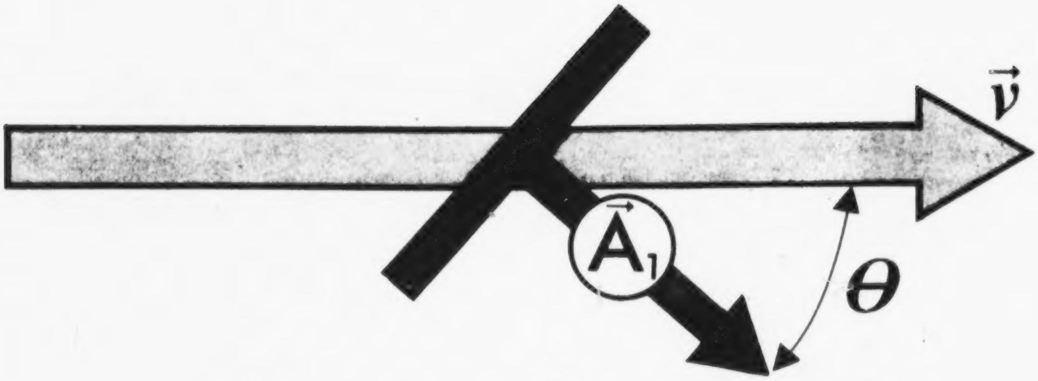

$\vec{v}$
$\vec{A}_1$
θ

FIGURE ⑤

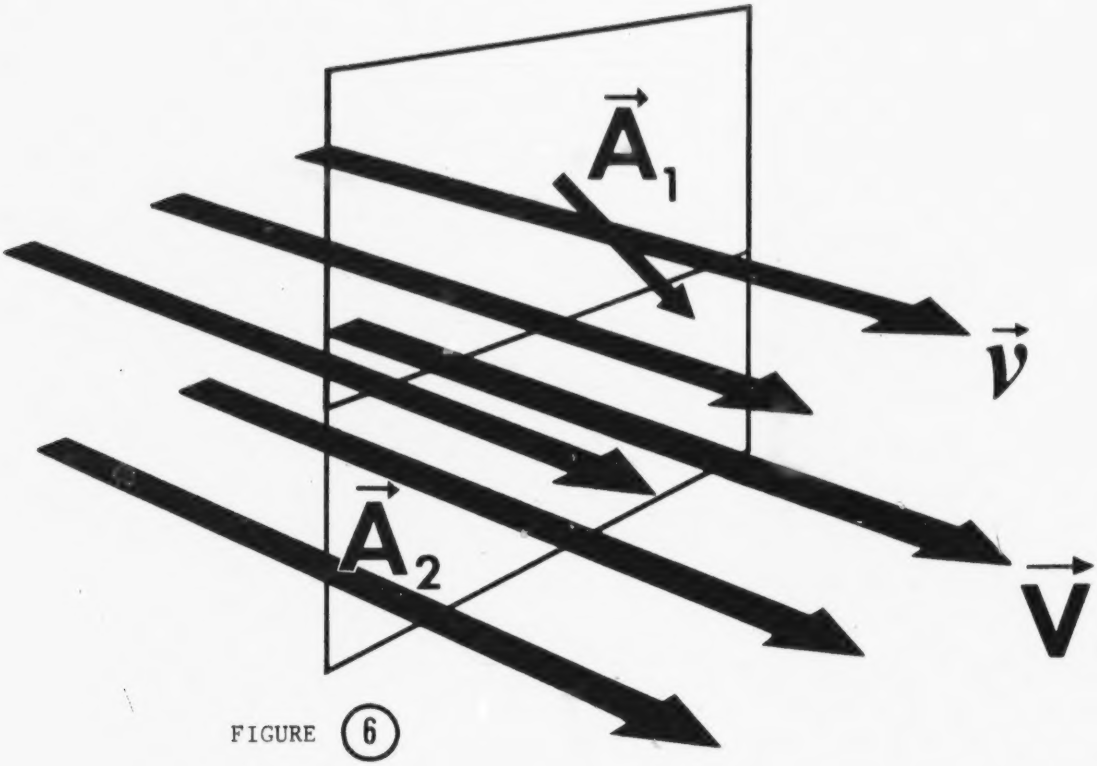

$\vec{A}_1$
$\vec{v}$
$\vec{A}_2$
$\vec{V}$

FIGURE ⑥

$$\phi = \vec{v} \cdot \vec{A}_1 + \vec{V} \cdot \vec{A}_2$$

FIGURE ⑦

(Figure 8) Consider a general case involving an arbitrary surface in a velocity field. An element of area $d\vec{A}$ is arbitrarily selected on the irregular surface and the velocity at this element of area is taken as $\vec{v}$. By definition, the flux through this element of area is

$$\phi = \vec{v} \cdot d\vec{A}$$

(Figure 9) To find the flux through the entire surface, it is merely necessary to sum up the individual fluxes through the elemental areas over the entire surface.

(Figure 10) The correct expression for the required integration is shown in this figure. The integral is a surface integral and the integration process includes the entire area.

(Figure 11) The discussion thus far has been based on a velocity field in which the flux has been evaluated in terms of volume of fluid per unit time through a given surface. Since the same general approach may be utilized when an electric field is substituted for the velocity field, in this figure the electric field vector $\vec{E}$ has been used to replace the velocity vector $\vec{v}$. A similar pattern of thinking results in the definition of electric flux through an element of area as the surface integral of the dot product of the electric vector and the element of area.

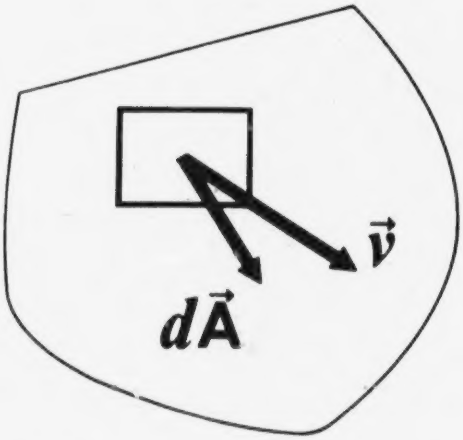


FIGURE ⑧

$$\phi = \sum \vec{v} \cdot d\vec{A}$$

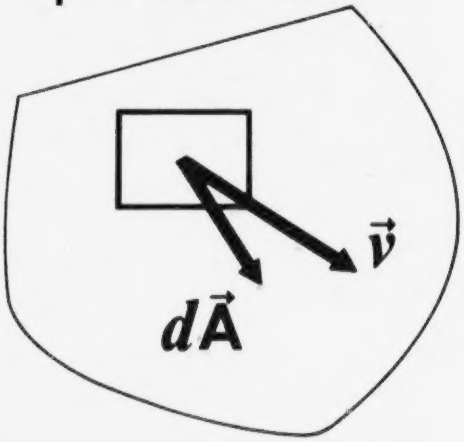


FIGURE ⑨

$$\phi = \int \vec{v} \cdot d\vec{A}$$

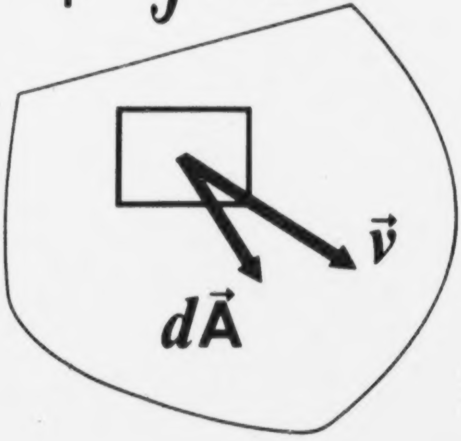


FIGURE ⑩

$$\phi = \int \vec{E} \cdot d\vec{A}$$

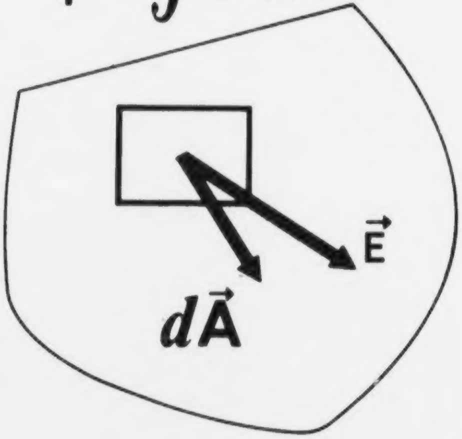


FIGURE ⑪

(Figure 12) A typical example is represented in this figure. It is desired to calculate the total flux through the designated area shown. The length L of the rectangle lies alown the positive x-axis, and the width of the rectangle lies in a plane that forms an angle of 30° with the xz plane. In addition, it is assumed that the electric field vector passes through the surface in the positive y-direction.

(Figure 13) The electric field is not uniform. As indicated, the electric intensity is given by the relation $\vec{E} = \vec{a}z$, showing that the magnitude of the field is a function of the z-coordinate. Thus, $\vec{E}$ will vary from zero at $z = 0$ to infinity when z = infinity. All of this is preliminary information.

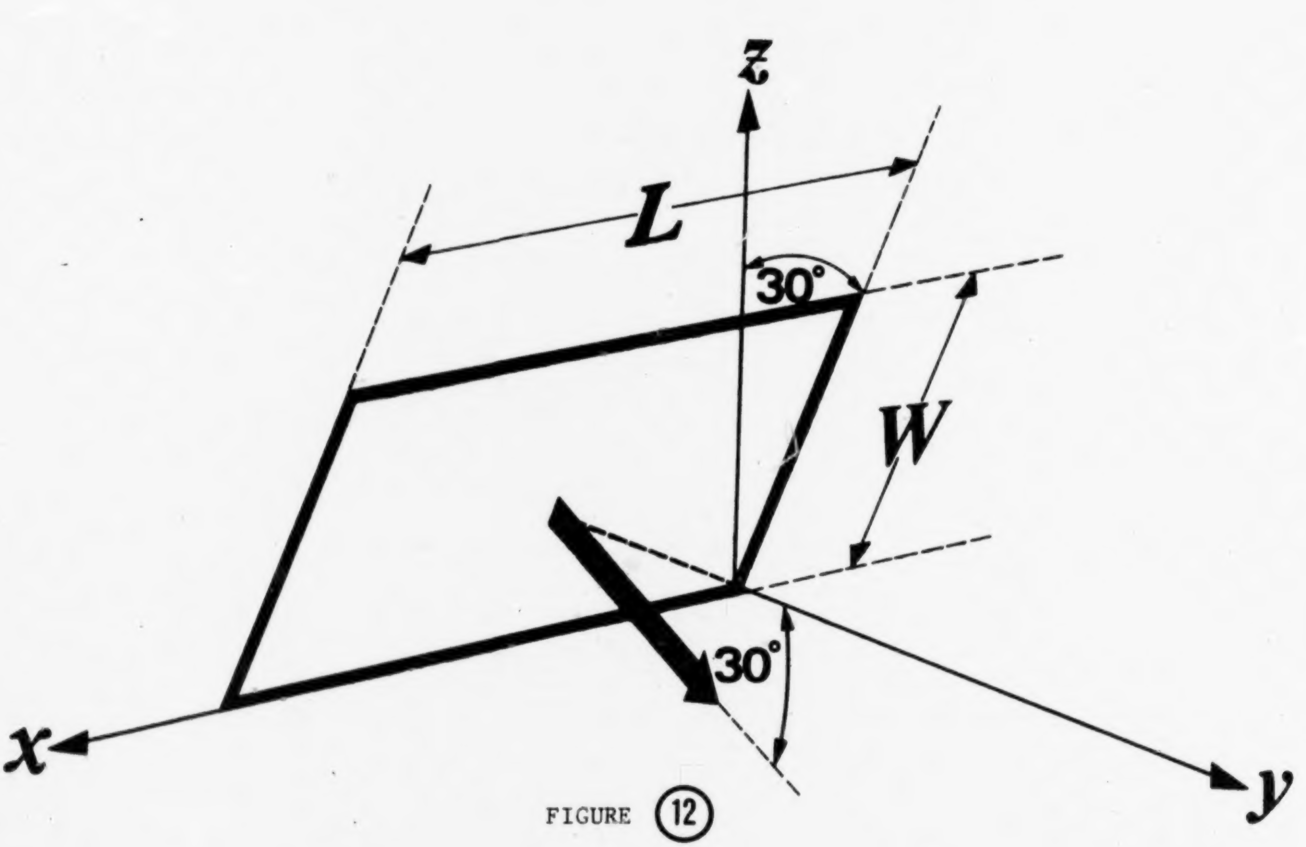


FIGURE ⑫

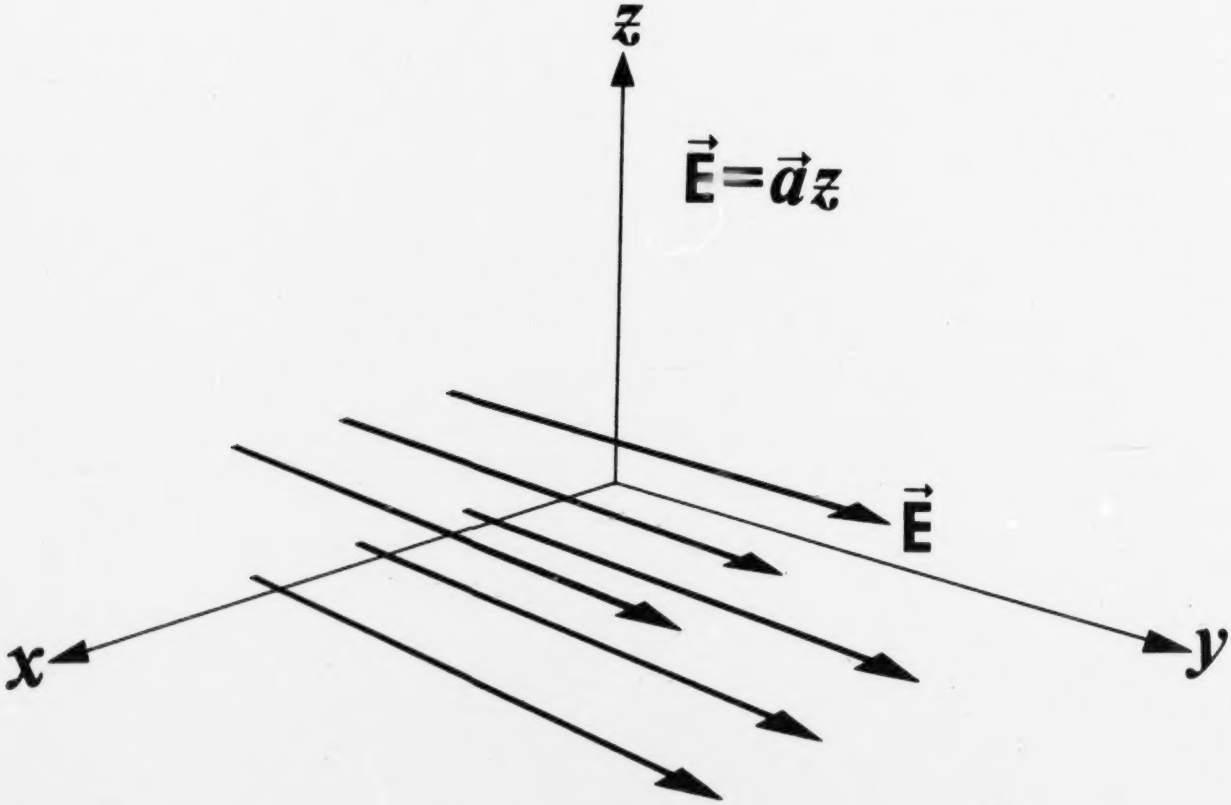

z
$\vec{E} = \vec{a}_z$
$\vec{E}$
x
y

FIGURE ⑬

(Figure 14) At some distance from the lower edge of the rectangle, a thin strip of width dw and length L is selected. The area of the strip is, of course, L dw. The next step involves the representation of this small area by a vector as illustrated in the figure, directed at an angle of 30° downward toward the xy-plane. To calculate the flux through this element of area, it is necessary to find the value of $\vec{E}$ at this distance from the xy-plane.

(Figure 15) To determine $\vec{E}$, the z-coordinate at the distance w from x-axis must now be determined. A perpendicular is dropped from dw to the xz-plane so that the z-coordinate of dw is seen to be w cos 30°. Since the electric intensity E is the product az, the intensity at the distance w is $E = a\,w\cos\theta$.

(Figure 16) This figure shows the step-by-step development of the expression for evaluating $d\phi$ for the conditions specified. The student is urged to study the sequence carefully until he is certain of complete comprehension.

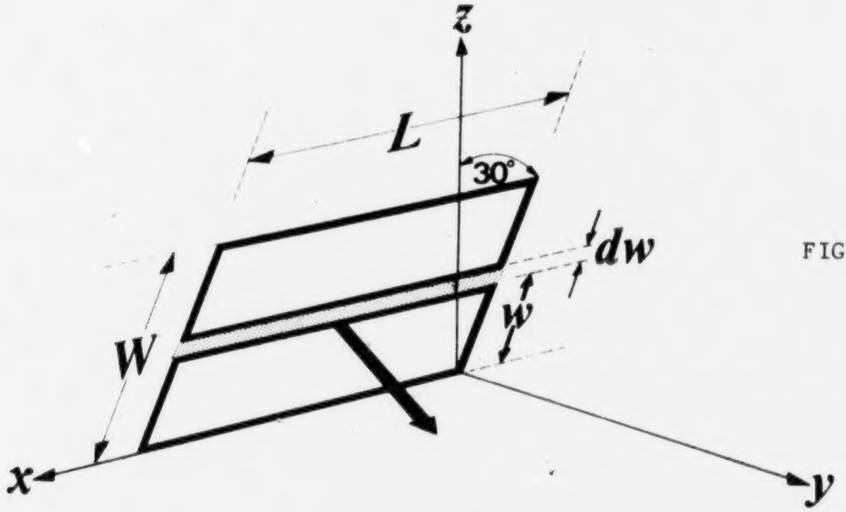

z
L
30°
dw
w
W
x
y

FIGURE ⑭

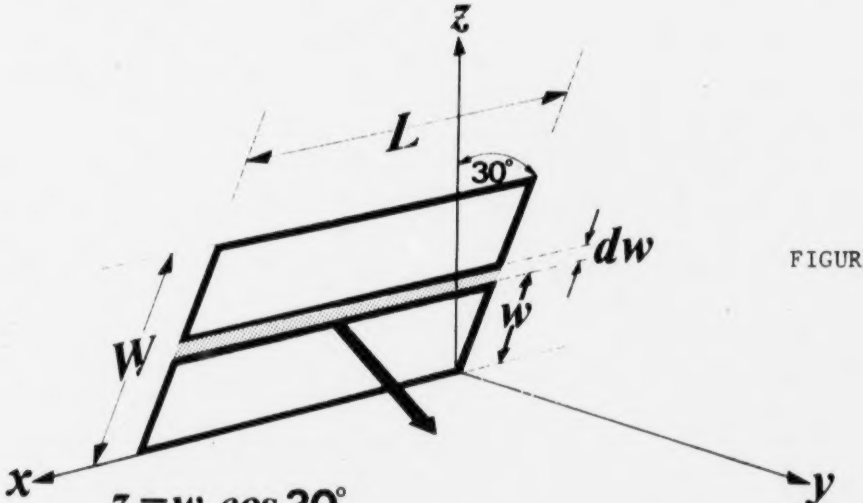

z
L
30°
dw
w
W
x
y

FIGURE ⑮

$z = w \cos 30°$

$\vec{E} = \vec{a} z$

$= a w \cos 30°$

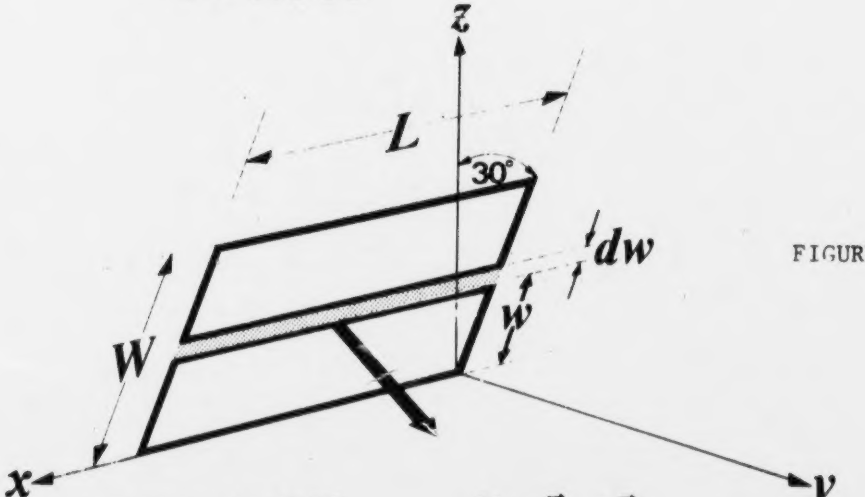

z
L
30°
dw
w
W
x
y

FIGURE ⑯

$z = w \cos 30°$

$\vec{E} = \vec{a} z$

$= a w \cos 30°$

$d\phi = \vec{E} \cdot d\vec{A}$

$= E L \, dw \cos 30°$

$= a w L \cos^2 30° \, dw$

(Figure 17) The expression for the flux is here shown in integral form.

(Figure 18) In this solution, the integrand can be simplified substantially as shown so that the final expression for the flux is

$$\phi = L\ a \cos^2 30^\circ\ \frac{W^2}{2}$$

The procedure described in this paper is sufficiently general so that it may be applied to various problems encountered in the calculation of flux.

$$d\phi = awL\cos^2 30^\circ dw$$

$$\phi = \int_0^W awL\cos^2 30^\circ dw$$

FIGURE ⑰

$$d\phi = awL\cos^2 30^\circ dw$$

$$\phi = \int_0^W awL\cos^2 30^\circ dw$$

$$= \tfrac{1}{2} aW^2 L\cos^2 30^\circ$$

FIGURE ⑱

FLUX

TERMINAL OBJECTIVES

10/1 A Answer questions and solve problems concerning electric field flux.

CALCULATION OF $\vec{E}$ USING GAUSS' LAW

Gauss' Law is an extremely powerful tool for calculating the electric field due to continuous charge distributions. Two specific configurations are discussed in this paper. Both of these involve charge distributions possessing a high degree of symmetry. The first case to be examined concerns itself with the electric field due to an infinitely long wire with a continuous charge distribution. If the linear charge density on the wire is λ, the length of the wire L will contain a charge given by the expression

$$q = \lambda L$$

where q is the total charge on the length L of the wire. It is important to note that L is not the entire length of the wire, but merely a segment of an infinitely long wire, and that for an infinitely long wire, the field E is everywhere perpendicular to the length of the wire. (Figure 1)

The second case to be considered is the field due to an infinite sheet of charge. In this case, one is interested in the electric field E at some distance from this infinite sheet of charge. The figure shows a finite sheet of area A. As is the case with the long wire, this sheet is merely a portion of an infinite sheet of charge. The reader is asked to recall that the surface density of charge is the charge per unit areas, usually symbolized by σ (sigma). The total charge contained on the section of area A will then be given by the expression.

$$q = \sigma A$$

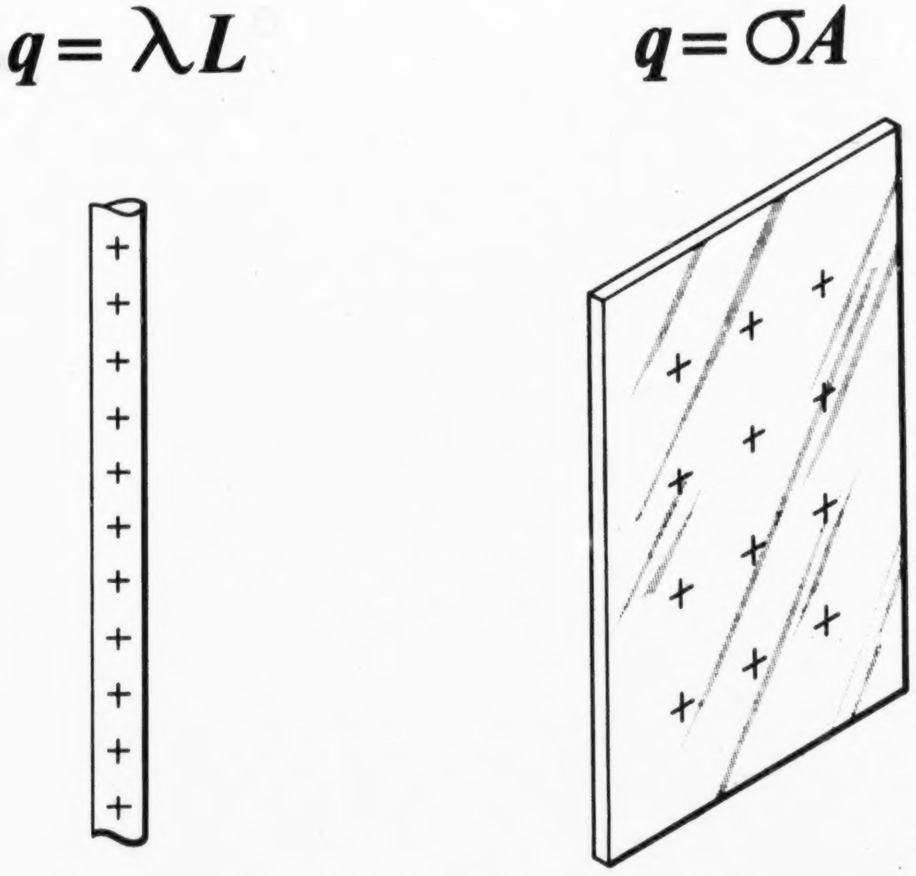

$q = \lambda L$
$q = \sigma A$
+

FIGURE ①

A very important point must be repeated here. In the following analysis, the wire of length L and the plate of area A represent finite sections of an infinite wire and an infinite plate respectively. In the case of the sheet of charge, the reader should recall that the field due to a positive sheet will always be directed perpendicularly away from the plane of the sheet.

(Figure 2).

At this point the stratgey to be used in the application of Gauss' Law will be discussed. This strategy is the same for each case. The first step is to draw a closed symmetrical surface around the charge. This closed surface is usually called a Gaussian surface. The second step is to apply Gauss' Law which states that the flux through any closed surface containing a charge q is given by

$$\phi = \frac{q}{\varepsilon_0}$$

(Figure 3)

The third step in the procedure makes use of the definition of electric flux. This definition states that the flux is given by

$$\phi = \int \vec{E} \cdot d\vec{A}$$

taken over any area A. In the problems to be considered here, however, the use of Gauss' Law provides a symmetrical Gaussian surface that surrounds the charge.

STRATEGY

(1) *Draw a closed symmetrical surface around the charge*

(2) *GAUSS' LAW*
$\phi = q/\epsilon_o$

FIGURE ②

STRATEGY

(1) *Draw a closed symmetrical surface around the charge*

(2) *GAUSS' LAW*
$\phi = q/\epsilon_o$

(3) $\phi = \int \vec{E} \cdot d\vec{A}$

FIGURE ③

(Figure 4)

Because of the high degree of symmetry of the Gaussian surfaces, the above integral in the definition of flux generally reduces to EA.

(Figure 5)

Returning to the infinite wire, the reader is reminded that the segment of wire has a length L and a linear charge density λ. The problem is to calculate the electric field E at the point P, which is a distance r from the segment of wire.

The first step is the construction of a symmetrical surface surrounding the charge. A reasonable Gaussian surface for this charge distribution is shown in figure 6.

This Gaussian surface is a cylinder whose axis coincides with the axis of the wire. Let the cylinder have a radius r and a length L. It would be in the best interests of the reader to note this construction very carefully.

The second step is the application of Gauss' Law which may be stated as follows: the flux through the Gaussian surface is $\frac{q}{\varepsilon_o}$. It has been shown that the total charge q on the length L of the wire is λL, thus the flux through the Gaussian surface must be $\frac{\lambda L}{\varepsilon_o}$.

$$\phi = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA$$

FIGURE ④

CHARGE DENSITY = λ

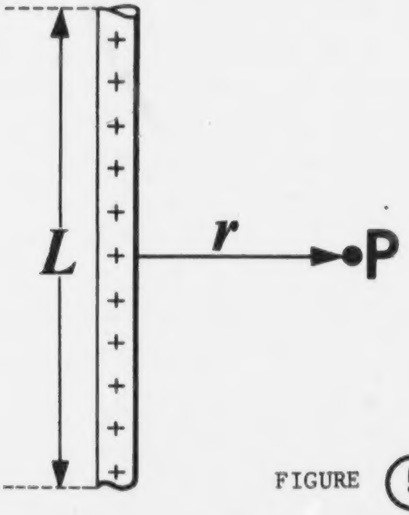
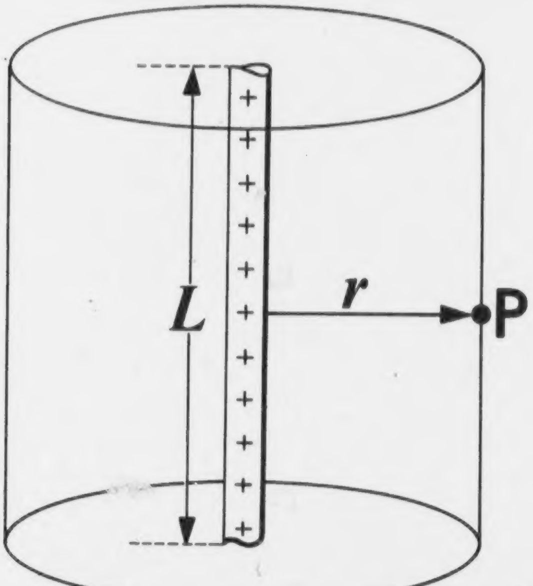


FIGURE ⑤

CHARGE DENSITY = λ



$$q = \lambda L$$
$$\phi = \lambda L / \epsilon_o$$

FIGURE ⑥

(Figure 7)

The third step of the strategy makes use of the fact that the electric flux through the cylinder is the product of the magnitude of the electric field E, and the surface area of the wall of the cylinder.

The reader may wonder why the top and bottom of the cylinder are not considered. The reason for this is that since the electric field lines are always perpendicular to the wire, they will always be parallel to the end caps of the cylinder. If the lines of E are parallel to the end caps of the cylinder, they do not pass through them and therefore need not be included in the analysis.

(Figure 8)

Since the flux equals EA, where A is the area of the cylinder wall, the flux may be stated in terms of the radius of the cylinder or

(1) $\phi = 2\pi rLE$

Now all one need do is equate equation (1) with the definition of flux, namely $\phi = \frac{q}{\varepsilon_o}$

However, recall that the charge q on the length L of the wire is

$q = \lambda L$

so that the equation for flux becomes

(2) $\phi = \frac{\lambda L}{\varepsilon_o}$

Equating equations (1) and (2), one obtains

(3) $2\pi rLE = \frac{\lambda L}{\varepsilon_o}$

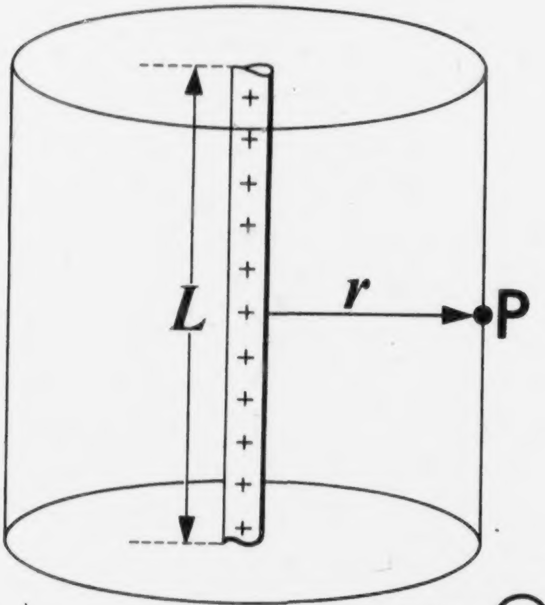
Solving equation (3) for E yields

$E = \frac{\lambda}{2\pi\varepsilon_o r}$

as the result.

The reader is reminded that this conclusion has already been readied by another method. That method involved an integration. It should be clear that the method using Gauss' Law provides a much simpler approach to the problem than does the method of integration.

CHARGE DENSITY= λ



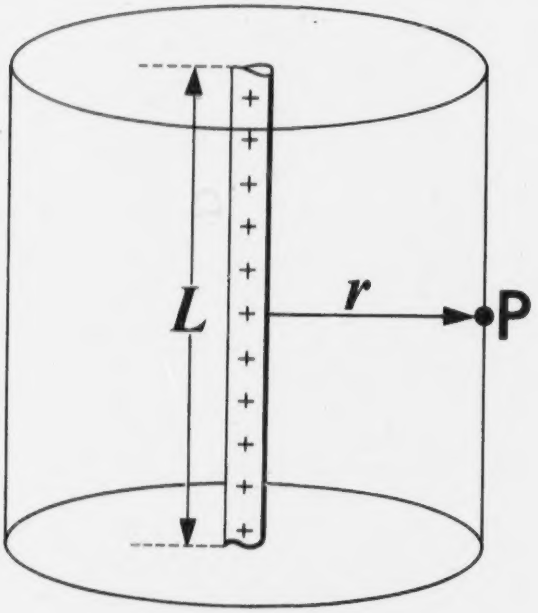
$$q = \lambda L$$
$$\phi = \lambda L/\epsilon_o$$

$$\phi = EA$$
$$= E\,2\pi r L$$

FIGURE ⑦

CHARGE DENSITY= λ

$$E = \frac{1}{2\pi\epsilon_o}\,\frac{\lambda}{r}$$



$$\phi = EA$$

(1) $\phi = 2\pi r L E$

$$\phi = q/\epsilon_o$$

$$q = \lambda L$$

(2) $\phi = \lambda L/\epsilon_o$

(3) $2\pi r L E = \lambda L/\epsilon_o$

$$E = \frac{\lambda}{2\pi\epsilon_o r}$$

FIGURE ⑧

(Figure 9)

The second case will now be considered. In this configuration there is a flat plate of cross sectional area A and charge density σ. Note that this is a surface charge density, i.e. charge per unit area. The charge on the plate is assumed to be positive. At a point P at any distance from the plate, the electric field E is directed perpendicularly away from the plate. The analysis to be used here is similar to the one used above for the case of the wire.

(Figure 10)

In the first step, a Gaussian surface must be constructed around the plate. For a plate, the best Gaussian surface is a parallelepiped whose end faces are parallel to the surface of the plate.

(Figure 11)

For this step it is important to note that the total charge on the surface is the product of charge density and area or

(4) $q = \sigma A$

Recall that the flux is given by

(5) $\phi = \frac{q}{\varepsilon_o}$

If equation (4) is substituted into equation (5) one obtains

(6) $\phi = \frac{\sigma A}{\varepsilon_o}$

which is an expression for the flux in terms of surface charge density and area. Note that this is the flux passing out of the Gaussian surface drawn around the plate.

CHARGE DENSITY = σ

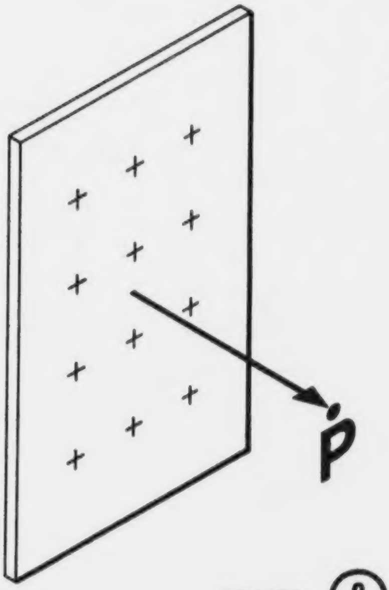


FIGURE ⑨

CHARGE DENSITY = σ

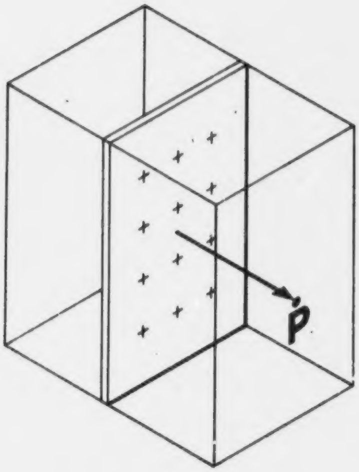
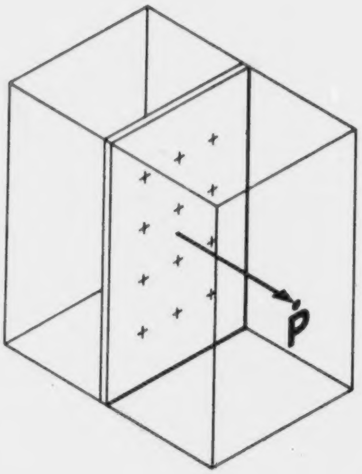


FIGURE ⑩

CHARGE DENSITY = σ



$$q = \sigma A$$

$$\phi = \frac{\sigma A}{\epsilon_0}$$

$$(4)\ q = \sigma A$$

$$(5)\ \phi = \frac{q}{\epsilon_0}$$

$$(6)\ \phi = \frac{\sigma A}{\epsilon_0}$$

FIGURE ⑪

(Figure 12)

Because of the high degree of symmetry of the configuration, Gauss' Law may be used to specify the outgoing flux. At this time, the exact meaning of A must be clarified. Since A is a flat plate, flux emanates from both sides. Thus, the area of consideration is not merely the area of one side of the plate, but the area of both sides. Hence, if the area of one side of the plate is A, the total area for the emanation of flux will be 2A. The expression for flux then becomes

$$\phi = 2\ EA \tag{7}$$

Equating the two expressions for flux in equations (6) and (7), one obtains

$$2\ EA = \frac{\sigma A}{\varepsilon_0}$$

and the result for the electric field strength is

$$E = \frac{\sigma}{2\varepsilon_0}$$

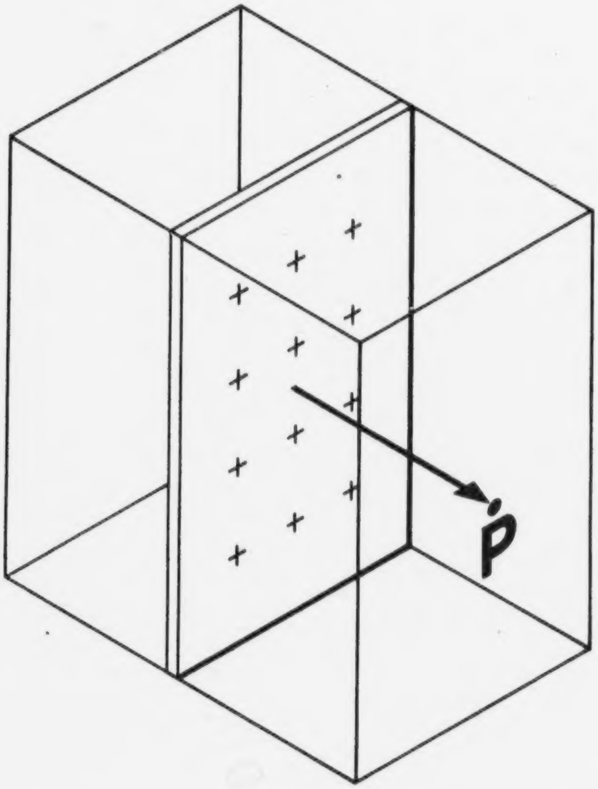
The two examples described above involve highly symmetric charge distribution in which Gauss' Law is clearly simpler than methods involving integration. This is particularly true in the case of the determination of the field due to an infinitely long wire in which there is a uniform distribution of charge.

The reader should consider both these cases carefully, particularly with respect to the strategy that has been used involving the three-step solutions shown. Once these are firmly established in his mind, he should then apply the same strategy to other symmetrical configurations.

CHARGE DENSITY $= \sigma$

$$q = \sigma A$$

$$\Phi = \frac{\sigma A}{\epsilon_0}$$



(4) $q = \sigma A$

(5) $\Phi = \frac{q}{\epsilon_0}$

(6) $\Phi = \frac{\sigma A}{\epsilon_0}$

(7) $\Phi = 2EA$

$$2EA = \frac{\sigma A}{\epsilon_0}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

FIGURE ⑫

CALCULATION OF $\vec{E}$ USING GAUSS' LAW

TERMINAL OBJECTIVES

10/2 A Answer questions and solve problems using Gauss's Law for cases of spherically symmetric charge distributions.

10/2 E Apply Gauss' Law to charged bodies.

I 517

CAPACITORS

Capacitors are found by the score in virtually every kind of electronic device, performing many different and important duties. Yet, regardless of the nature of the task handled by a specific capacitor, its usefulness may be traced to its ability to store an electric charge and deliver this charge in the form of a potential difference or an electric current when called upon to do so. Capacitors have much in common with ordinary mechanical storage devices such as jugs, bottles, and tanks. The capacity of a bottle to hold fluid, for example, is in many respects analogous to the capacitance of a capacitor. Perhaps the best way to introduce the significance of electrical capacitance is to start with one such analogy.

Figure 1 shows two balloons identified as A and B; each one has been partially inflated. At first glance, there is a strong temptation to say that B has the greater capacity for air because it has a larger inflated volume than A. But if both balloons are now deflated, suppose that they then appear as in Figure 2. Assuming that both are fabricated of the same rubber material, it would then appear that A should have the greater capacity to hold air because it is larger than B initially. This is quite true, but the fact remains that it is quite easy to inflate the smaller balloon to a larger inflated volume merely by using more air pressure on it than on the other balloon. This, of course, was done in obtaining the result in Figure 1. Balloon B, despite its smaller initial size, was inflated to larger size than A simply by blowing harder on it. This is very much like comparing a gallon jug with a quart bottle; although the gallon jug has a greater capacity than the quart bottle, you can if you wish put a pint of water into it so that it contains less water than the filled quart bottle with its smaller capacity. In short, the capacity of a balloon or a bottle is not at all the same thing as the actual quantity of air or liquid that it may happen to be holding at any given instant.

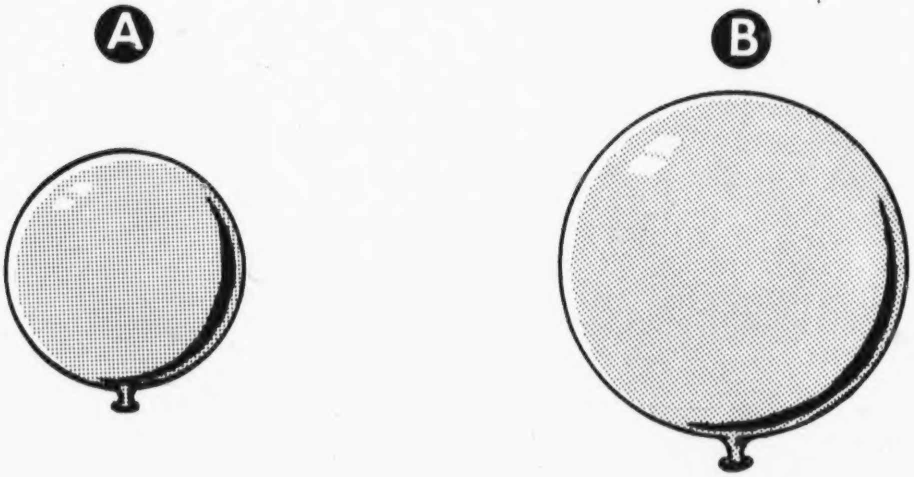


FIGURE ①

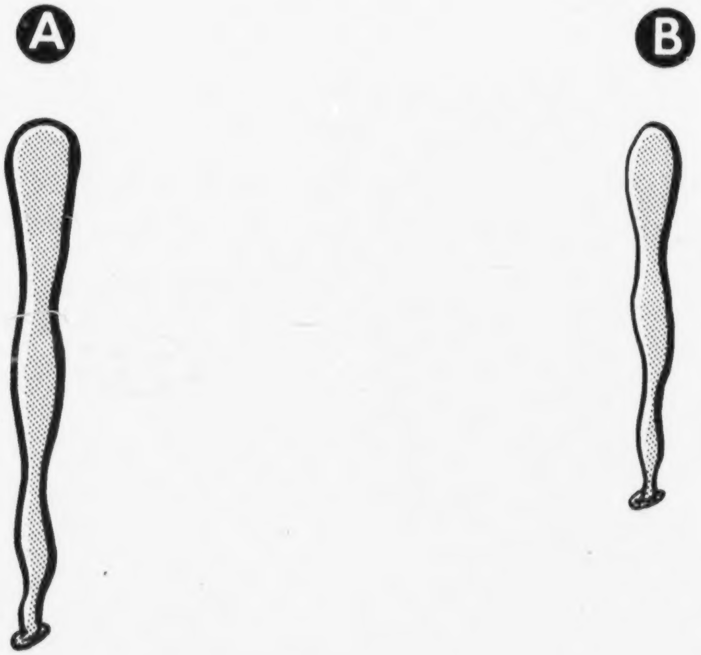


FIGURE ②

The analogy can be carried further. The same two balloons, A having a larger capacity than B, are connected to a common source of air pressure through a T-tube as shown in Figure 3. This arrangement insures that the same air pressure will be applied to both balloons. Furthermore, let it be assumed that when the balloons have expanded somewhat they then develop sufficient back pressure to equalize the pressure of the source. At this point, inflation will cease and the air system will be in equilibrium. The result is shown in Figure 3. Here it is seen, as might have been anticipated, that the balloon of larger capacity -- balloon A -- has grown to a larger size than balloon B. The quantities of air in the two balloons have been designated as Q_A and Q_B, respectively. It is also not unreasonable to guess that Q in either case would be directly proportional to the capacity of the respective balloons. If one has twice the initial size or capacity of the other than it ought to be able to hold twice the quantity of air when the pressure is the same for both. If the capacities are called C_A and C_B, then the expressions shown in Figure 4 would apply. Pressure, being constant, may be taken as the constant of proportionality so that the equal-ratio form may be written as $Q = PC$. Or, in the final form, capacity may be considered to be defined as the ratio of quantity of air to the magnitude of the air pressure used to inflate the balloon. Perhaps a better "feel" for the significance of this defining expression can be realized by putting it this way:

(1) If balloon A can hold a greater quantity of air at a given pressure than balloon B, it must have a correspondingly greater capacity. That is, C varies directly with Q when P is constant.

(2) If a greater pressure is required to bring balloon B up to the same volume of air as balloon A, then the capacity of balloon B must be smaller. That is, C varies inversely with P when Q is constant.

Thus, capacity may be defined as quantity per unit pressure.

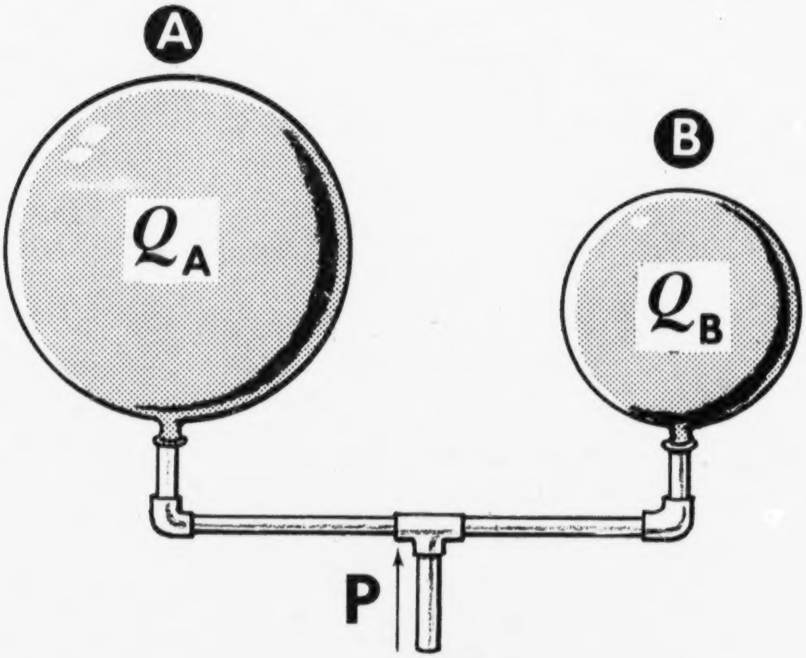


FIGURE ③

WITH PRESSURE CONSTANT

$$\frac{Q_A}{C_A} = \frac{Q_B}{C_B}$$

OF $Q = PC$

SO $C = \frac{Q}{P}$

FIGURE ④

The word "capacitance" rather than "capacity" is now in common use in electricity. However, the capacity of a balloon is very closely analogous to the capacitance of a capacitor. Two capacitors, C_1 and C_2, are illustrated in Figure 5. Let it be assumed that the materials and the method of fabrication used in the construction of both of these were identical. Since C_1 is physically larger than C_2, it is reasonable to conjecture that the former would be capable of holding a larger electrical charge than the latter. Q is again used to denote "quantity", this time quantity of electrical charge. Refer now to Figure 6.

For simplicity, let C_1 be twice the capacitance C_2. If both are charged from the same source of potential -- and here potential difference is analogous to the air pressure used to fill the balloons -- then C_1 should accept twice the charge Q that will pass into C_2. So in this case, potential difference is taken as the constant of proportionality just as pressure was previously and the expression relating quantity of charge Q, capacitance C, and potential difference V appears as shown in Figure 6. Thus, since $C = Q/V$, capacitance may be defined as charge per unit potential difference. Referring to the form $Q = CV$, it is at once seen that the quantity of stored charge in a capacitor can be increased by increasing the capacitance at constant voltage, or by increasing the voltage at constant capacitance, or by increasing both voltage and capacitance simultaneously.

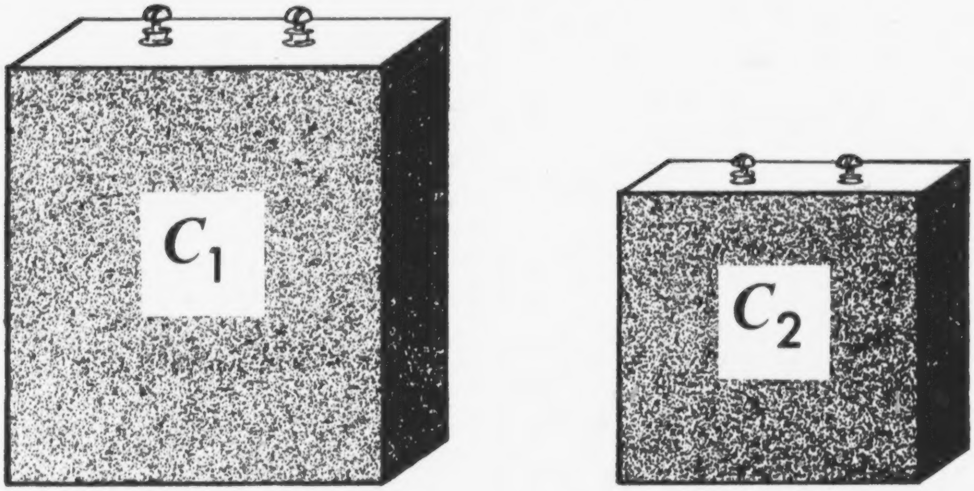


FIGURE ⑤

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} \quad (V = k)$$

$$\text{so} \quad C = \frac{Q}{V}$$

FIGURE ⑥

The simplest form of capacitor comprises a pair of parallel conducting plates separated by a vacuum as illustrated in Figure 7. The capacitance of this device is given by the ratio of the charge on either plate to the potential difference between the two plates. The remainder of this discussion will be devoted to the derivation of an equation in which capacitance for this simple capacitor is expressed in terms of its dimensions A and d, plate area and plate separation respectively.

After charging, each of the capacitor plates as shown in Figure 8 has accepted a charge of Q. The left plate (purely arbitrarily, of course) has a charge of -Q and the right plate a charge of +Q. Since the area of each plate is A, the charge density on the surface of each plate is then Q/A. The lower-case Greek sigma is generally used to denote surface charge density so, as in Figure 9, the density condition is $-\sigma$ on the left plate and $+\sigma$ on the right plate.

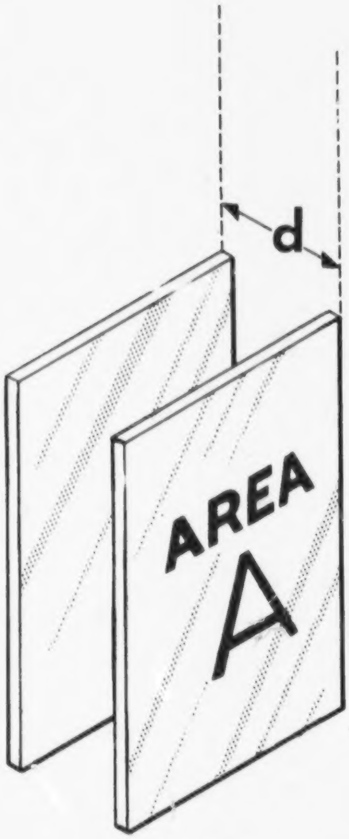


FIGURE ⑦

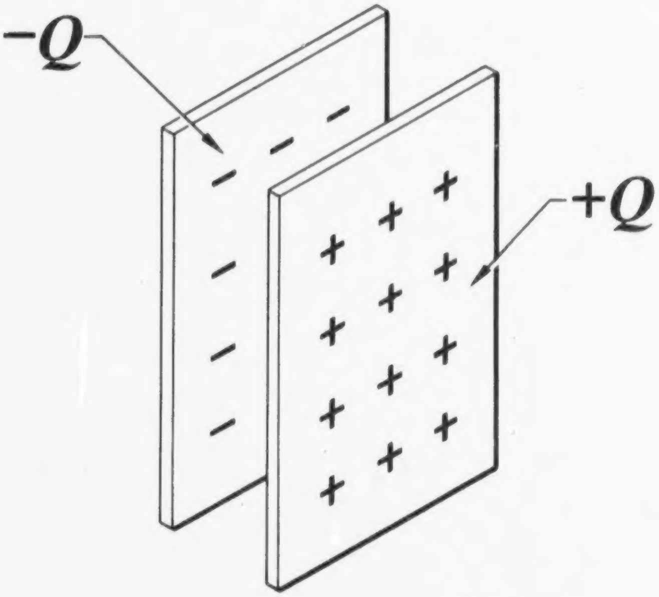


FIGURE ⑧

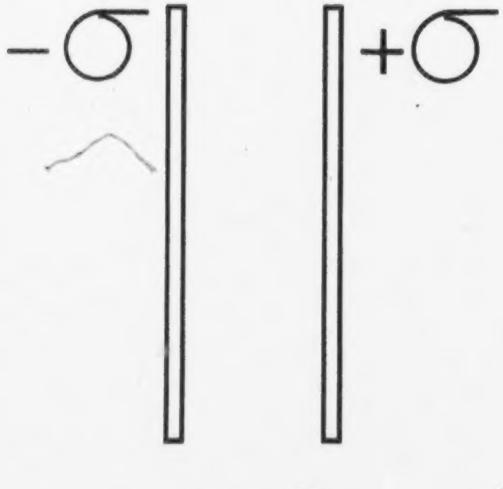
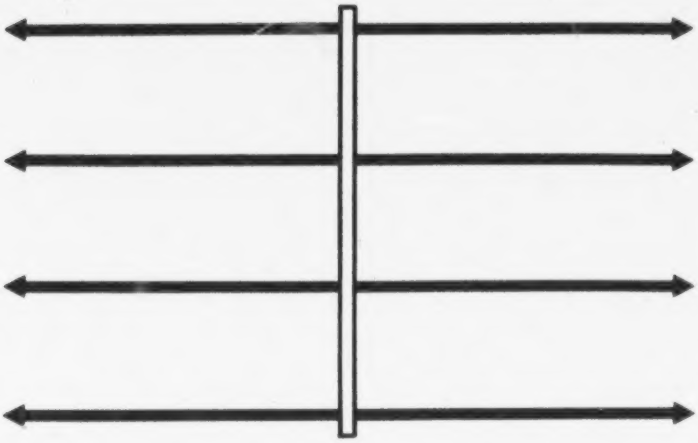


FIGURE ⑨

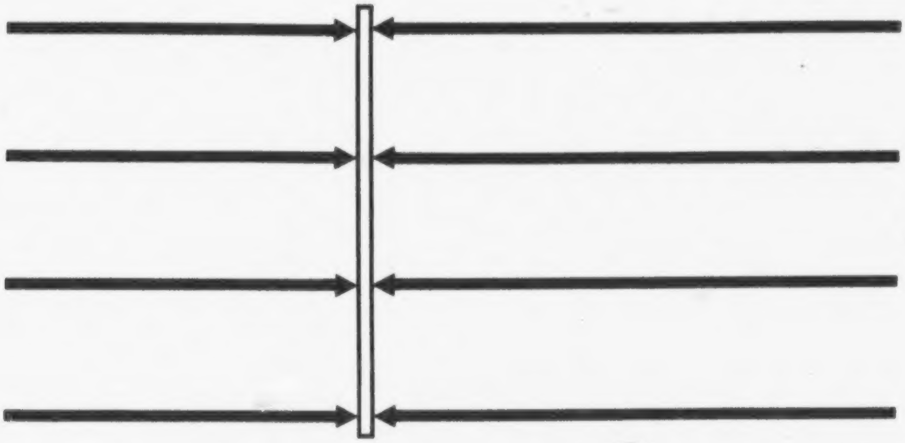
Considering only the positive plate for a moment, there will be an electric field outside both surfaces having the general form illustrated in Figure 10. This field is directed away from both surfaces at right angles and has the magnitude shown. The negative plate has a field of exactly the same magnitude but opposite in sign indicating that its field is directed toward the conducting surface instead of away from it. Refer to Figure 11.

In combination, the situation changes as follows: since the field outside either plate is a net field due to two equal fields oppositely directed, these outside fields cancel out completely. Between the plates, however, the fields due to each plate are similarly directed, hence the resultant field is the sum of the two individual fields. This means that between the plates the electric intensity is expressed as shown in Figure 12, and is directed from the positive to the negative plate.



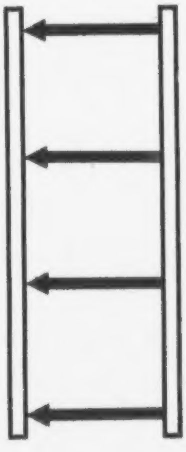
$$E = \frac{+\sigma}{2\epsilon_0}$$

FIGURE ⑩



$$E = \frac{-\sigma}{2\pi\epsilon_0}$$

FIGURE ⑪



$$E = \frac{\sigma}{2\epsilon_0} + E = \frac{\sigma}{2\epsilon_0}$$

$$E = \frac{\sigma}{\epsilon_0}$$

FIGURE ⑫

Using this expression for the electric field intensity between plates, the potential difference between the plates can now be evaluated. Refer to Figure 13. Potential difference is the integral of E.dl from zero to d separation. When this integration is performed, the result is V = Ed. Then, when σ/ε_0 is substituted for E, the final expression in this figure is obtained.

Figure 14 shows the two equalities that can be used to progress to the objective of this discussion. The first is a statement of the potential difference V in terms of one dimension of the capacitor, the distance separating the plates. The second describes the value of the charge Q in terms of the area A of either plate. This second equation is merely an algebraic conversion of the definition of surface charge density so that it is seen that total charge may be expressed as the product of charge per unit area (sigma) and the total area.

Before going on to Figure 15, the student should make the necessary substitutions in C = Q/V to obtain the final simplified expression for C in terms of A and d. Figure 15 shows how this is done, the final equation being

$$C = \varepsilon_0 \frac{A}{d}$$

Since ε_0 is a constant -- it is called the permittivity of a vacuum -- then the capacitance of any capacitor of the type discussed here is dependent only on the area A of one of its plates and the distance between the plates. In other words, capacitance is not influenced by potential differences nor circuit connections. A capacitor may be labeled by the manufacturer purely on the basis of its dimensions, nothing else.

$$E = \frac{\sigma}{\epsilon_o}$$

$$V = \int_o^d E \cdot dl$$

FIGURE ⑬

$$= E\,d$$

$$= \frac{\sigma}{\epsilon_o}\,d$$

$$V = \frac{\sigma}{\epsilon_o}\,d$$

FIGURE ⑭

$$Q = \sigma A$$

$$C = \frac{Q}{V}$$

FIGURE ⑮

$$= \frac{\sigma A}{\sigma d/\epsilon_o}$$

$$= \frac{\epsilon_o A}{d}$$

CAPACITORS

TERMINAL OBJECTIVES

11/3 A Answer questions and solve numerical problems involving the physical significance and units (basic and submultiples) of capacitance, C.

11/3 D Solve problems involving various conductor-pair geometries' and the corresponding capacitances.

12/1 A Solve discriptive and numerical problems involving capacitors in series and parallel combinations. (Note: All interconnecting wires are resistanceless).

12/1 D Predict the effect of adding a dielectric of known dimensions and material to a vacuum capacitor in both descriptive and quantitative situations.

THE CAPACITOR
IN ACTION

THE CAPACITOR IN ACTION

The subject of the discussion of this paper deals with the factors that govern the capacitance of a capacitor. The simplest form of capacitor consists of a pair of parallel conducting plates separated by an insulator or dielectric. The quantities involved in capacitor action are the magnitude of the charge transferred to it, the electric potential difference across its terminals, and its ability to store electric charges or capacitance. In the demonstrations to be described, a charge of constant magnitude will be considered to be present in the capacitor while variations of capacitance are studied in terms of changing potentials.

(Figure 1) The relatively crude instrument shown in the illustration resembles the basic instrument used in this discussion for the measurement of electric potential. A metallically coated styrofoam ball is suspended by means of a silk thread on a vertical metal stand held upright on a thick plastic insulating stand. If there are no electrical charges present, the ball hangs limply against the stand as in diagram A. When a negatively-charged is brought into contact with the top of the stand, some of the charges are transferred to the metal and become distributed throughout the conductor. The conductive coating of the styrofoam ball assumes a similar charge and is repelled by the stand. The extent to which the ball swings outward might be measured by the angle between the thread and the stand; in its equilibrium condition, where the gravitational force and electrical force are balanced, the angle might, if desired, be calibrated in terms of the electric potential responsible for the deflecting force. Transferring charge into the apparatus requires that work be done against the charges already present and, since potential is work per unit charge, the magnitude of the angle θ indicates qualitatively whether the potential is larger or smaller than it was for some other value of θ.

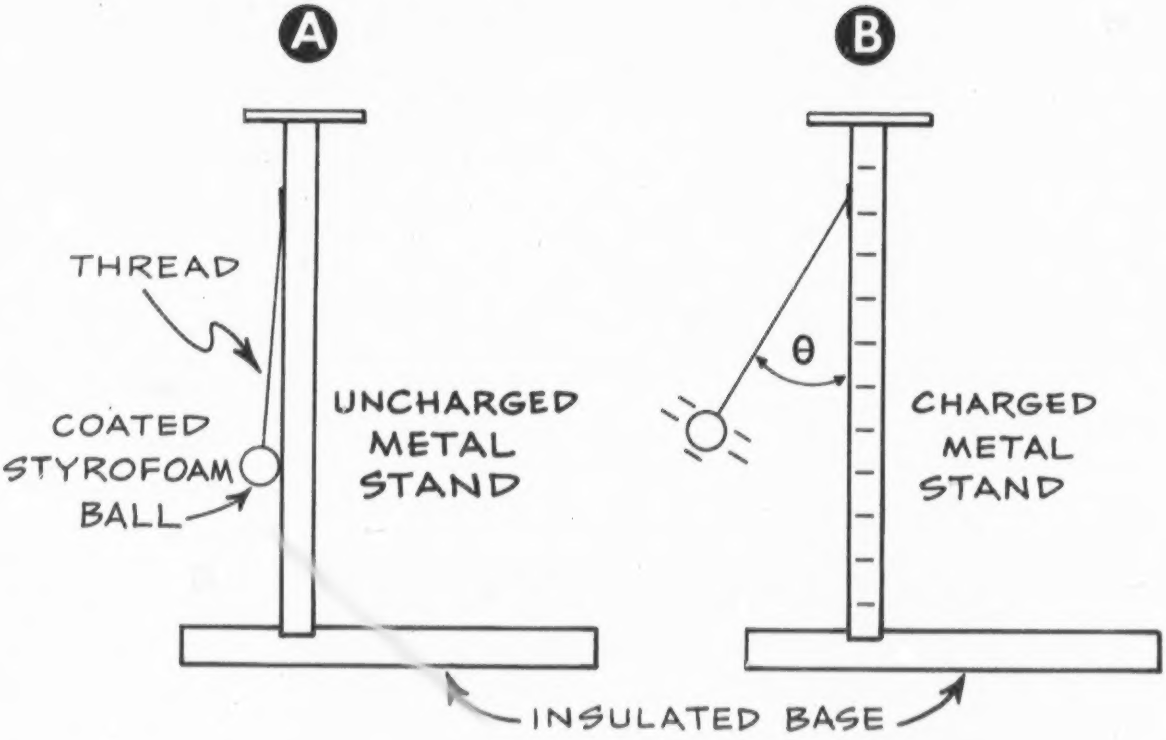

A
B
THREAD
COATED STYROFOAM BALL
UNCHARGED METAL STAND
θ
CHARGED METAL STAND
INSULATED BASE

FIGURE ①

(Figure 2) The Braun type of electroscope shown in this illustration is modeled after the simple arrangement just described. The active portion of this instrument is the aluminum vane pivoted near the center of the heavy metal support bar. Connection is made to the vane and support bar via a metallic path up to the aluminum terminal disc at the top. Note the large plastic insulator which keeps the vane assembly isolated from its surroundings. A metal ring called the shield surrounds the vane assembly but is not in electrical contact with it. The heavy metal base, electrically connected to the shield, completes the Braun electroscope. The lower portion of the vane is made very slightly heavier than the upper portion so that the vane normally rests in a vertical position when the electroscope is uncharged.

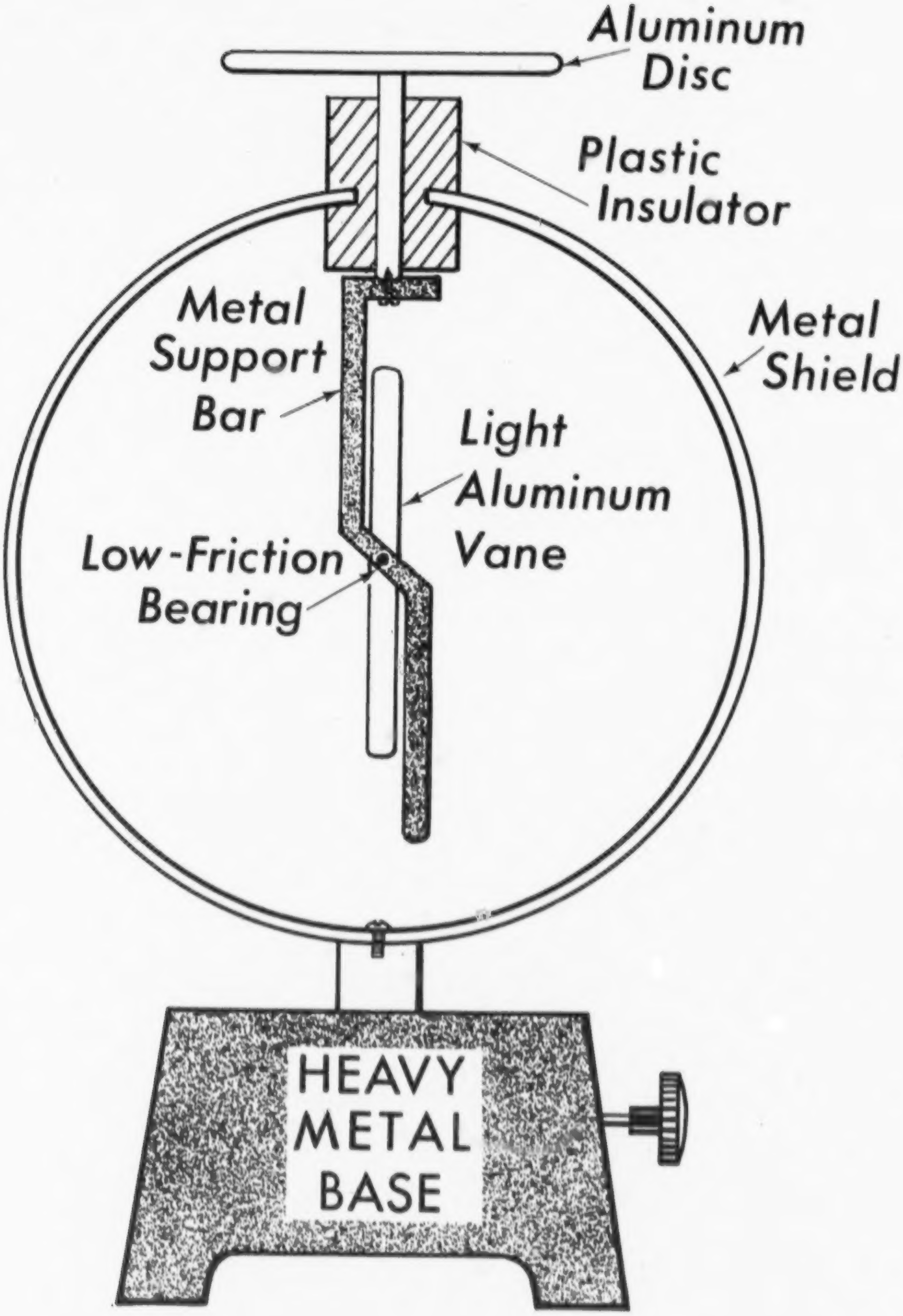
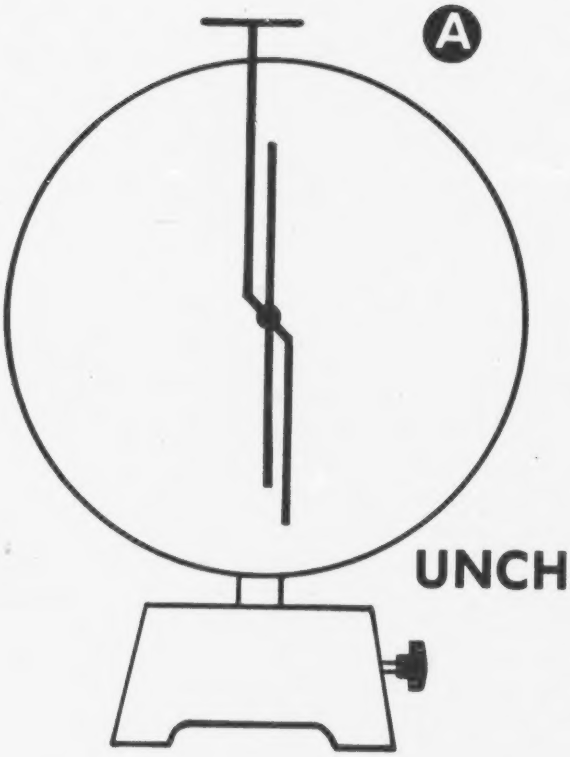
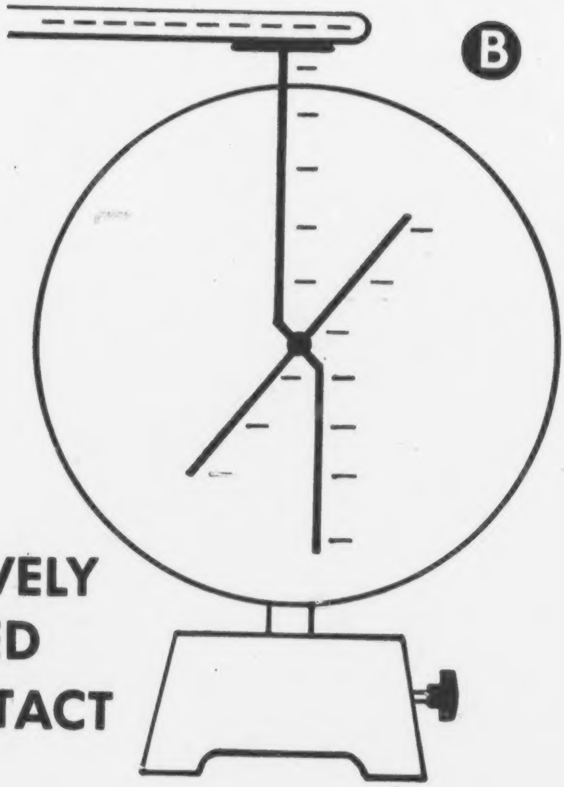

Aluminum Disc
Plastic Insulator
Metal Support Bar
Metal Shield
Light Aluminum Vane
Low-Friction Bearing
HEAVY METAL BASE

FIGURE ②

(Figure 3) Diagram A shows the electroscope in its neutral or uncharged condition with the vane vertical. In Diagram B, a negatively charged rod is brought into contact with the disc at the top of the electroscope, a part of the charge of the rod is transferred to the vane assembly so that both the support bar and the vane become similarly charged, and they repel once another. The couple acting on the vane then causes it to rotate to a new equilibrium position. The angle between the vane and the support bar may then be used as a measure of the potential difference between the vane and the shield. The shield is normally considered to be at the zero reference potential, or ground potential since it is in electrical contact with its environment. Thus, when reference is made to the potential on the vane assembly, it is to be understood that this potential is being observed or measured with respect to the shield. In the state shown in Diagram B, the electroscope is said to be negatively charged. A positive charge may similarly be transferred to the instrument by bringing it into contact with a positively charged body. When an electroscope is to be charged either positively or negatively, it is customary to start with the instrument in its uncharged state.



UNCHARGED



NEGATIVELY
CHARGED
BY CONTACT

FIGURE ③

(Figure 4) An insight into the mechanism of charge transfer may be gained by observing what happens when a charged body is brought into proximity with an electroscope already carrying a similar charge. In this illustration, the electroscope is initially charged negatively while a negative rod is brought close to the terminal disc. The initial negative charge potential is seen to be related to θ in Diagram A, with negative charges distributed more or less uniformly over the disc and vane assembly. Upon the approach of the negative rod, negative charges from the upper disc are forced downward into the vane assembly. This increases the charge density and hence the potential of the vane so that θ' now represents a measure of the new potential. Charges are not transferred in this case unless actual contact occurs; they are merely redistributed as a result of the increased electrical forces brought into play by the nearness of the heavily charged rod.

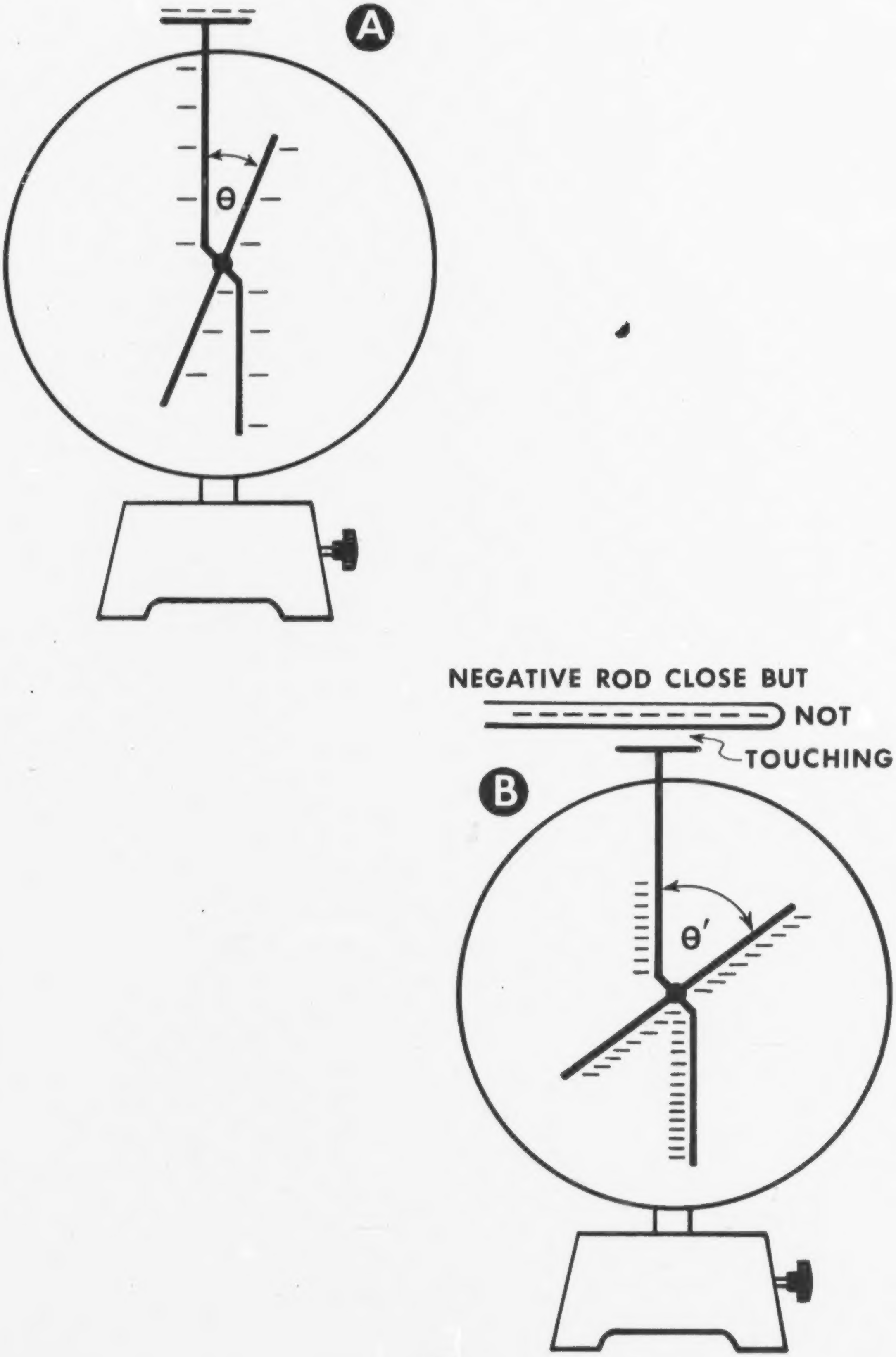

A
θ
NEGATIVE ROD CLOSE BUT NOT TOUCHING
B
θ′

FIGURE ④

(Figure 5) Here a positive rod has been brought close to the disc of a negative electroscope. In this case, negative charges are drawn away from the vane assembly by the coulomb attraction force causing a decrease in charge density in the vane and consequently a reduced potential. This is illustrated by the smaller value of θ'. A charged electroscope used as described in this Figure and in the previous one not only indicates the presence of charge on the approaching body, but also its polarity and comparative magnitude. As mentioned previously, voltage may be measured in terms of degrees of angle of deflection.

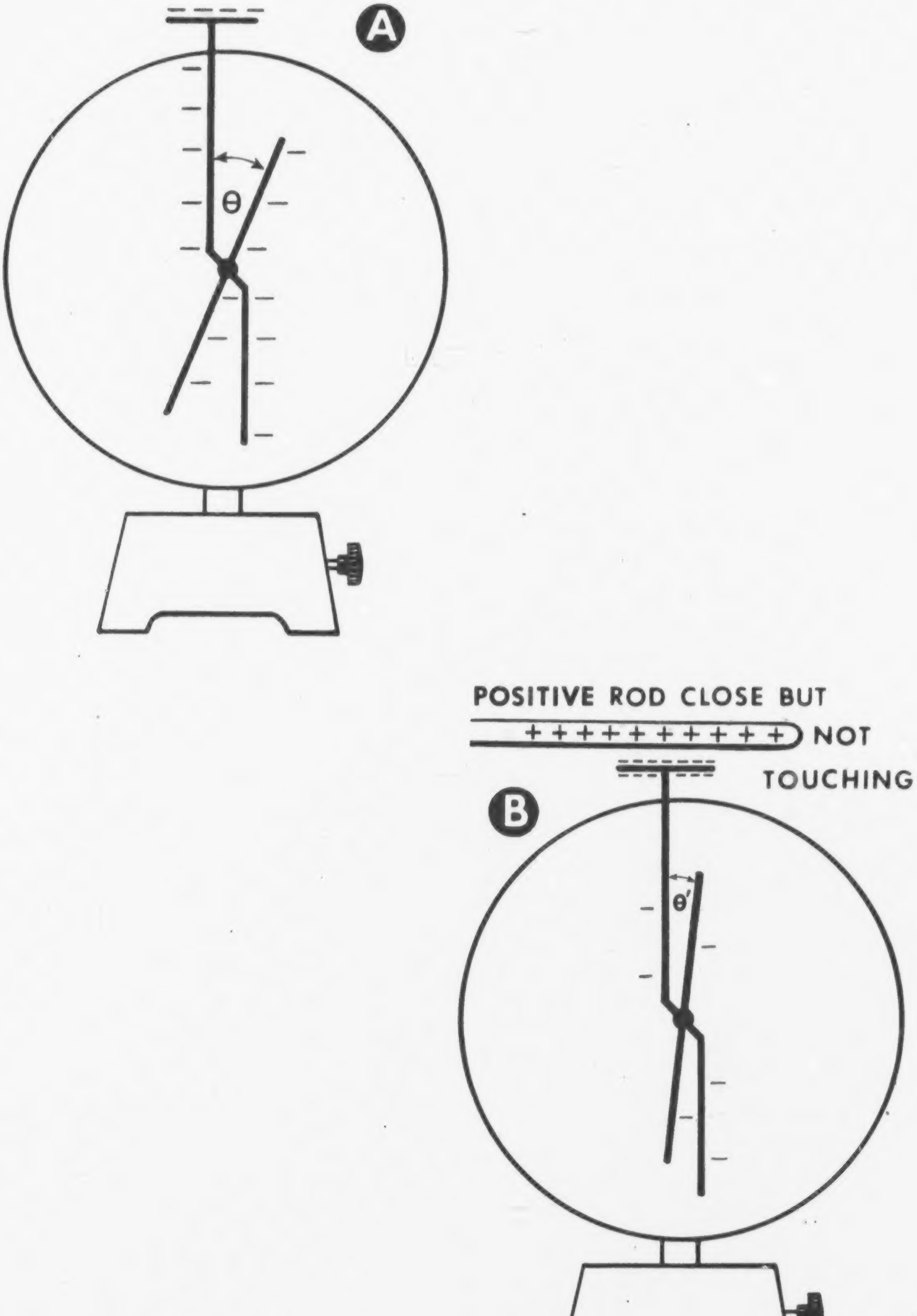

A
θ
POSITIVE ROD CLOSE BUT NOT TOUCHING
B
θ'

FIGURE ⑤

(Figure 6) A charged electroscope may be discharged or rendered neutral by touching the terminal disc with the finger. The conductivity of the human skin surface is sufficiently good to permit charges to be transferred to the body. Most of the charges on the Braun instrument will transfer to the human body, depleting the electroscope almost completely. Hence, the vane angle drops to zero when this is done. This method is universally used to discharge an electroscope in order to ready it for forthcoming tests.

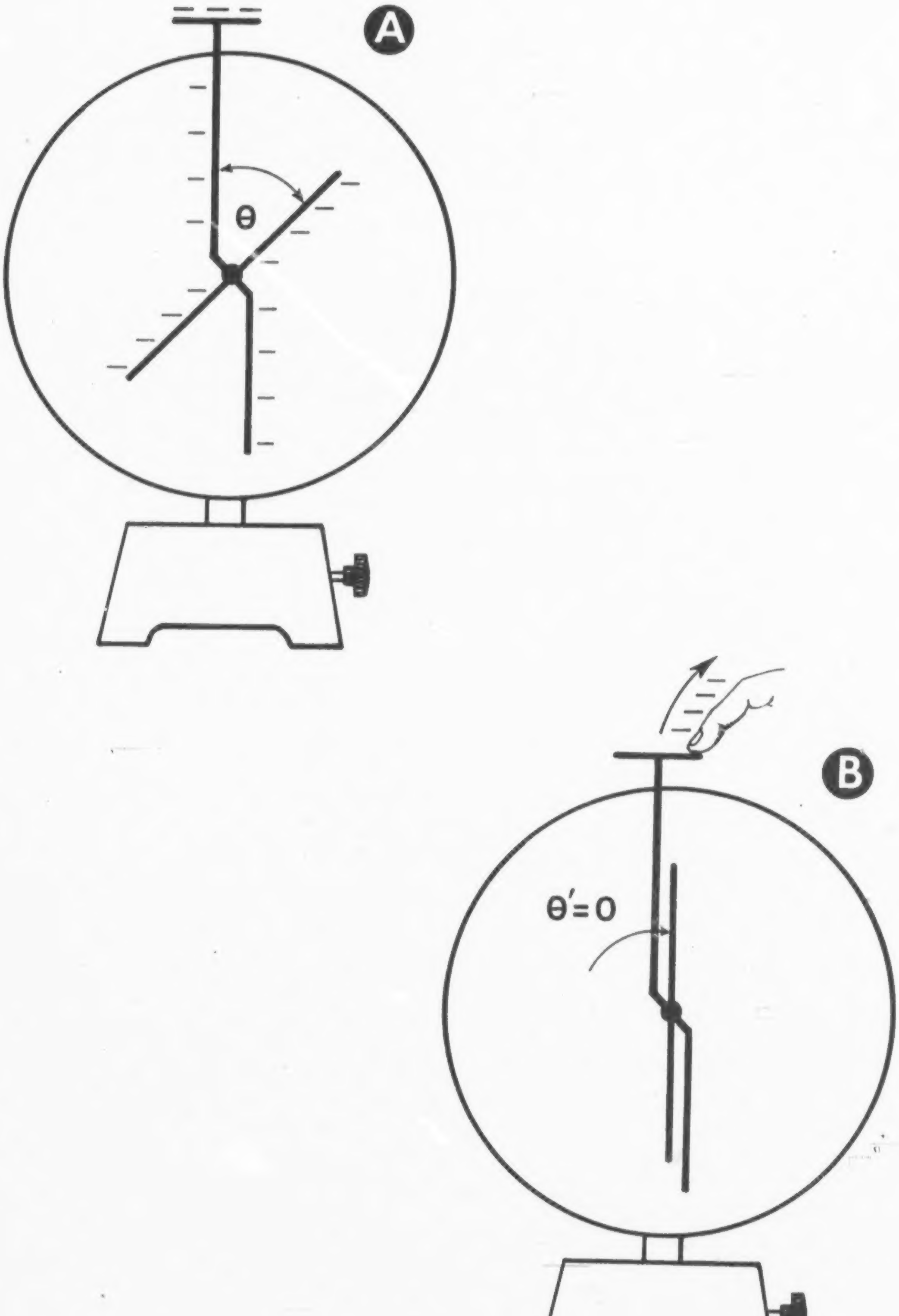
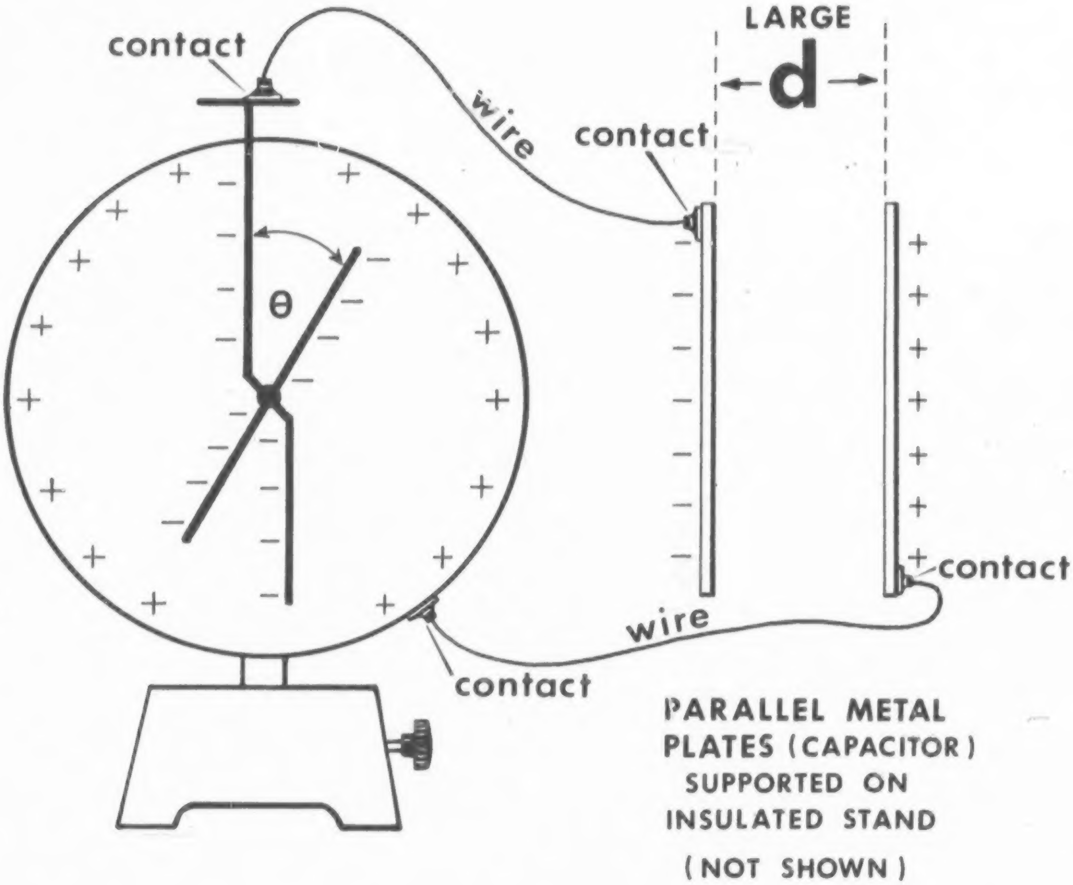


FIGURE ⑥

(Figure 7) The capacitor shown in this illustration is the simplest type, consisting of two parallel metal plates separated by air as a dielectric. The plates are isolated from the environment by an insulating stand which supports them in position. The left plate is connected by a wire to the terminal disc of the electroscope; the right plate is connected by a second wire to the shield and base assembly. The capacitor is then charged by stroking the left plate with a negative rod, positive charge of equal magnitude being induced in the right plate and in the shield assembly to which this plate is connected. The potential difference between the vane and the shield produces the deflection θ.

Equation 1 expresses the capacitance of a capacitor as the product of the dielectric constant K, the permittivity of empty space ϵ_o, and the ratio of plate area A to plate separation d. The Braun electroscope is now to be used to test this equation qualitatively. Equation 2 is a "tool" relationship which helps to confirm the results obtained by varying the quantities in Equation 1: the potential difference V between the capacitor plates is equal to the charge magnitude Q divided by the capacitance C. Note that the diagram indicates the separation distance d to be relatively large. In the next step, this distance is to be reduced while the vane deflection is observed.


contact
wire
LARGE
d
contact
θ
contact
wire
contact
PARALLEL METAL PLATES (CAPACITOR) SUPPORTED ON INSULATED STAND (NOT SHOWN)

$$(1) \quad C = K\epsilon_0 \frac{A}{d}$$

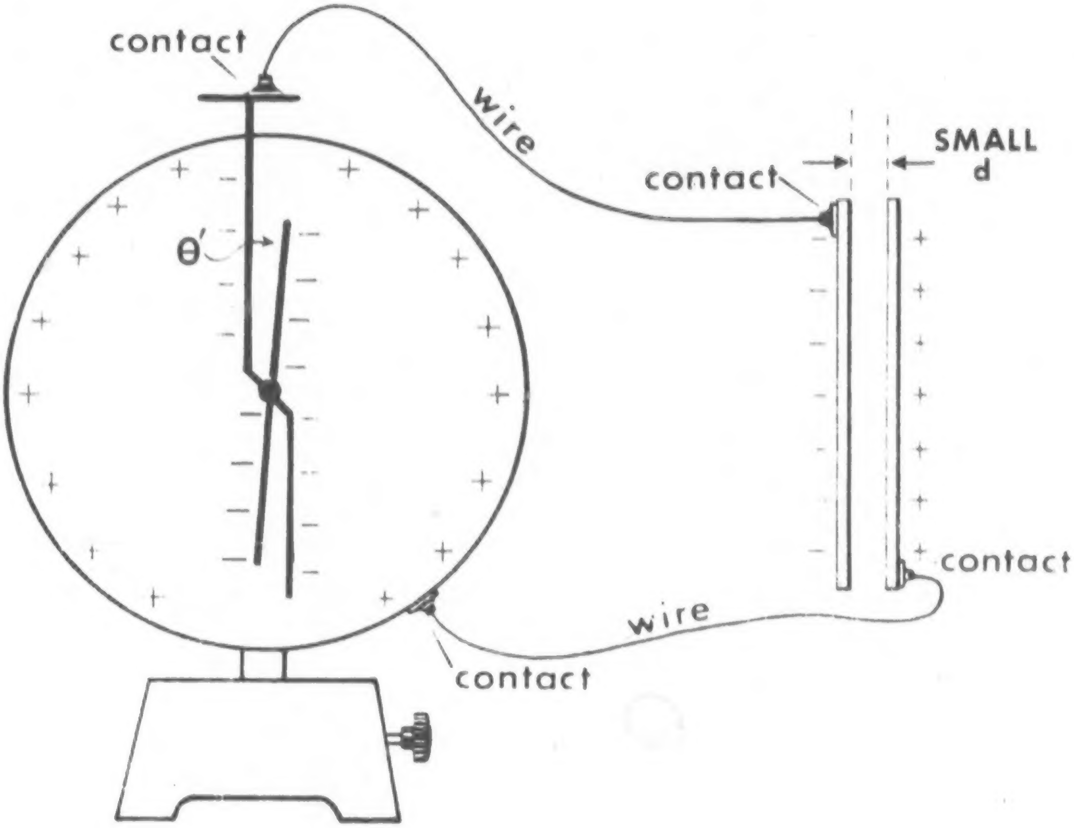
$$(2) \quad V = \frac{Q}{C}$$

(V is measured by θ)

FIGURE ⑦

(Figure 8) When the separation distance d is decreased as shown, it is at once observed that the deflection angle is correspondingly decreased with θ' being much smaller than θ. The equations predict this as indicated: when the initially-large d is reduced, the capacitance C must increase since C is inversely proportional to d. As a result, as shown in the second equation, the potential difference V between the plates must diminish because V is inversely proportional to C. Thus, the deflection angle goes from a large value (θ) to a smaller value (θ').

In the next step, the separation distance d is to be held constant while the plate area is raised from a small to a relatively large value.


contact
wire
contact
SMALL d
θ′
contact
wire
contact

The Changes

$$(1)\quad C = K\epsilon_0 \frac{A}{\mathbf{d}} \longrightarrow \mathbf{C} = K\epsilon_0 \frac{A}{d}$$

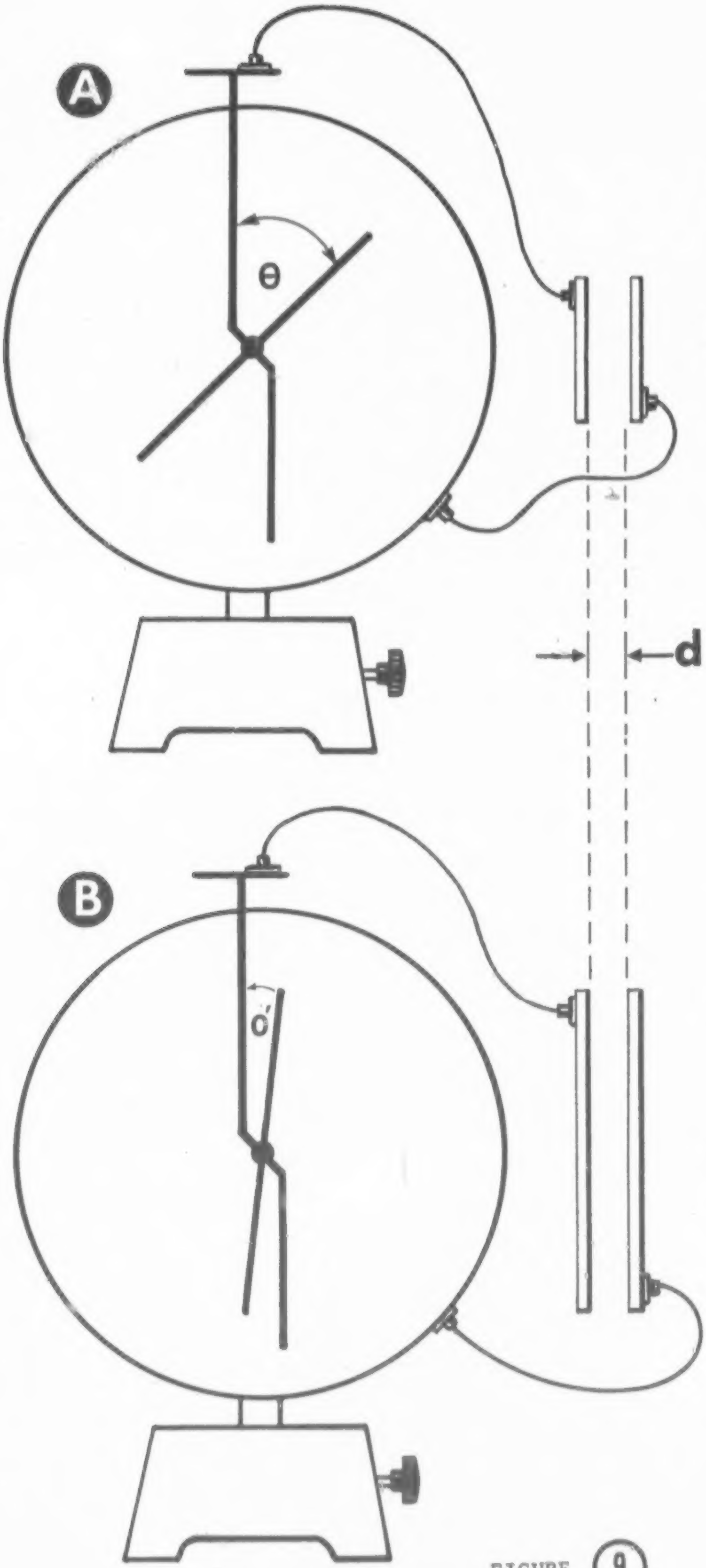
$$(2)\quad V = \frac{Q}{C} \longrightarrow v = \frac{Q}{\mathbf{C}}$$

$$\text{hence } \theta \longrightarrow \theta'$$

FIGURE ⑧

(Figure 9) As Diagram A shows, the initial area is relatively small; this has been indicated in Equation 1 by means of a small "A"; the value for θ obviously depends on the specific area, plate separation, and charge. In Diagram B, the area has been increased substantially, again causing the deflection angle to decrease. One may again reason predictively from the equations: As A becomes larger, C must also become larger because C is directly proportional to A. Again, as C grows, V must shrink correspondingly --- an inverse proportion exists here as has been stated previously. Thus, if V diminishes, the vane deflection must also decrease so that θ' is again smaller than θ.

Finally, the dielectric constant may be changed and the consequent change of capacitance noted.


A
θ
d constant
B
θ'

The Changes

(1) $C = K\epsilon_0 \frac{A}{d}$

(2) $V = \frac{Q}{C}$

θ = as shown

The Changes

(1) $C = K\epsilon_0 \frac{A}{d}$

(2) $v = \frac{Q}{C}$

θ' is smaller than θ

FIGURE ⑨

(Figure 10) The plate area A and the separation distance d are both to be held constant in this step. Diagram A illustrates that the deflection angle θ is relatively large so that a reasonably high potential is seen to exist across the capacitor plates. The air between the plates is serving as the dielectric. A sheet of polyethylene plastic is now inserted between the capacitor plates as in Diagram B; the deflection angle again decreases, indicating as before that the capacitance has increased. Equation 1 then shows that the insertion of the plastic material must have increased the dielectric constant K in order to increase the capacitance of the capacitor since these two quantities are directly proportional.

Equation 1 is thus verified qualitatively.

Consider a simple functional problem: a 12 microfarad capacitor of given dimensions and materials is restructured in such a way that its dielectric constant, plate area, and separation distance are all tripled in magnitude.

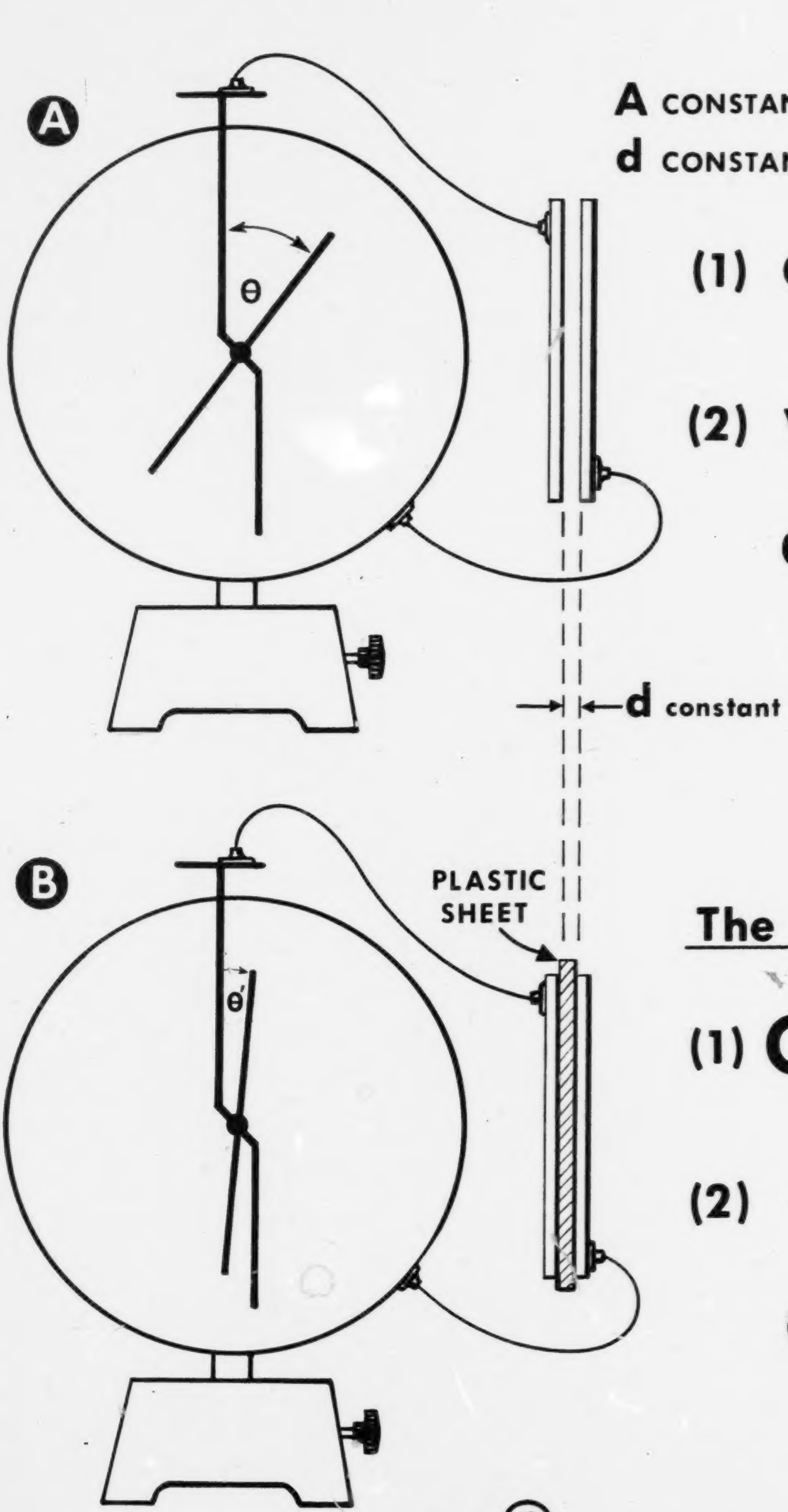
Don't turn to the next page before answering this question: what capacitance will the capacitor have after these changes have been made.

The solution follows:

$$12 \text{ ufd} = K \epsilon_o \frac{A}{d} \quad \text{(initially)}$$

$$? \text{ ufd} = (3K) \epsilon_o \frac{\cancel{3}A}{\cancel{3}d} \quad \text{(after restructuring)}$$

36 ufd = new capacitance.


A
θ
d constant
B
θ'
PLASTIC SHEET

A CONSTANT

d CONSTANT

(1) $C = K\epsilon_0 \frac{A}{d}$

(2) $V = \frac{Q}{C}$

θ = as shown

The Changes

(1) $C = K\epsilon_0 \frac{A}{d}$

(2) $v = \frac{Q}{C}$

θ' is smaller than θ

FIGURE ⑩

THE CAPACITOR IN ACTION

TERMINAL OBJECTIVES:

11-1.080-00

Solve descriptive and numerical problems involving capacitors in series and parallel combinations. (Note: All interconnecting wires are resistanceless)

11-1.083-00

Predict the effect of adding a dielectric of known dimensions and material to a vacuum capacitor in both descriptive and quantitative situations.

KIRCHHOFF'S RULES

We are now going to consider two rules that were first formulated by a physicist named Kirchhoff. These rules enable one to solve circuit problems, particularly in the case of complicated circuits. Such a circuit can be found in Figure 1.

FIGURE 1

The reader should note that the figure contains two seats of emf and several resistors. A typical problem might ask that the current through each resistor and the potential drop across each resistor be calculated, assuming you were given the emf's and the values of the resistors. Many problems of this nature can be solved by the method of equivalent resistors, but this method leads to very cumbersome algebraic exercises. Kirchhoff's Law can be of enormous help in this area, as it can eliminate much of the time consuming algebra.

The reader should direct his attention to Figure 2.

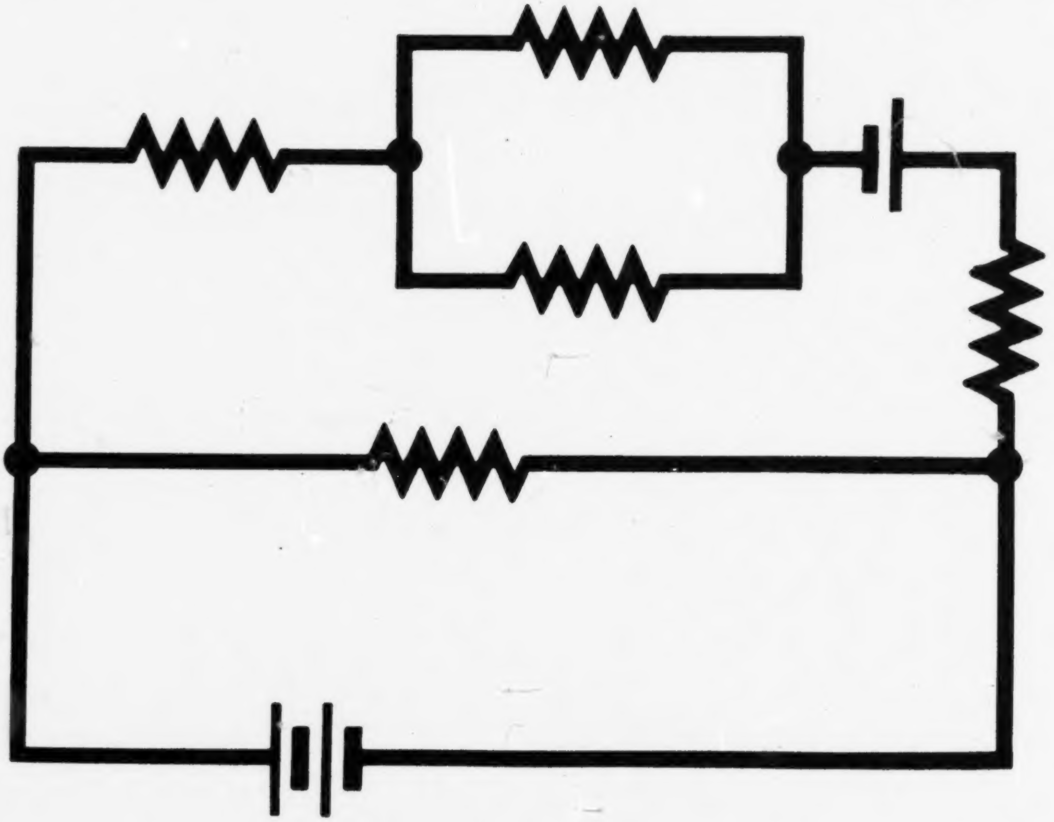


FIGURE ①

FIGURE 2

Before stating a formal definition of Kirchhoff's Laws, two definitions are in order. The first term to be defined is a junction. In Figure 2, the point that is labeled branch point or node is called a junction. That is, the three terms; branch point, node and junction all refer to the same idea. This author will refer to this point as a junction. A junction is defined as any point in a circuit at which the current can divide. For example, in Figure 2, the current i_1 divides at the junction into the currents i_2, i_3, and i_4. There is a very convenient convention for designating current entering the junction and current leaving the junction. Current entering a junction is taken to be positive and current leaving the junction is taken to be negative. Thus for the case shown in Figure 2, i_1 would be taken as positive and i_2, i_3, and i_4 would be taken as negative. Now let us give our attention to another concept, which is illustrated in Figure 3.

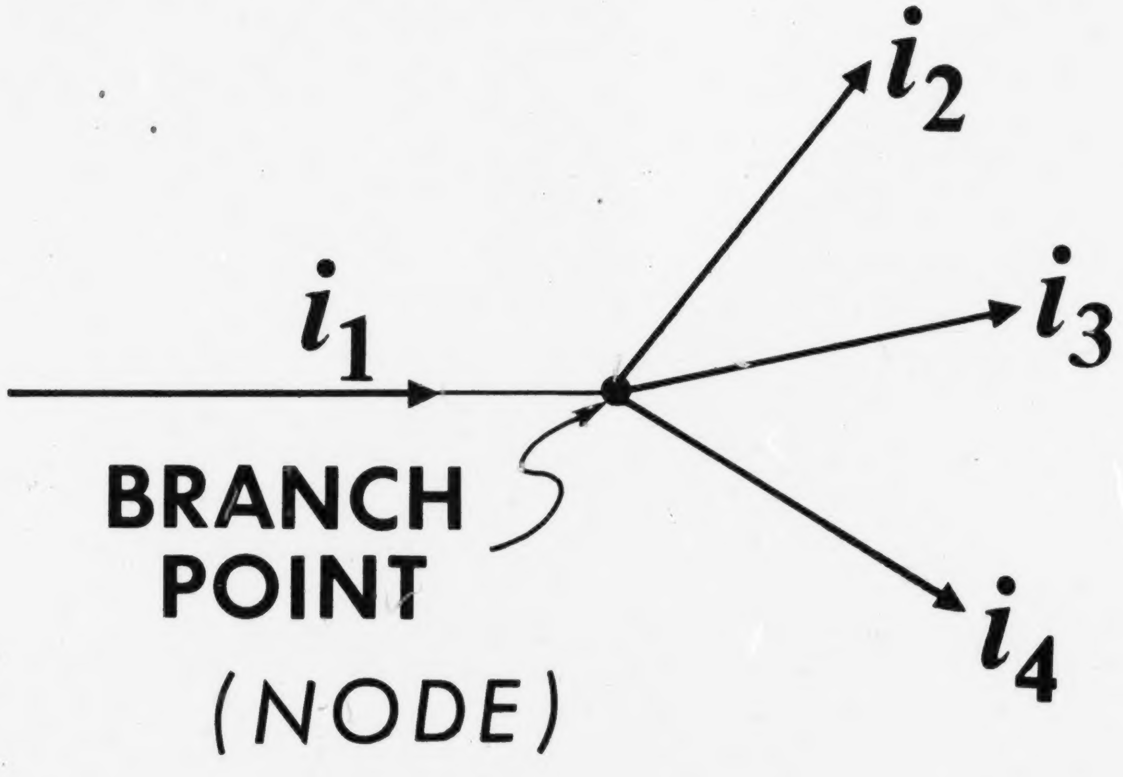

i_1
i_2
i_3
i_4
BRANCH POINT
(NODE)

FIGURE ②

FIGURE 3

This new concept is the loop. A loop is any closed conducting path in a circuit. In Figure 3, each of the dotted lines outlines a loop of the circuit. With these definitions, Kirchhoff's Laws may be formulated. The first of these rules is shown in Figure 4.

FIGURE 4

Kirchhoff's First Law may be stated as: At any junction, the algebraic sum of the currents must be zero. The question arises, "What does this mean from a practical viewpoint?" It means that the total current entering the junction must be equal to the total current leaving the junction. This rule may also be stated as: There can be no piling up of charge at the junction. The second of Kirchhoff's Laws is shown in Figure 5.

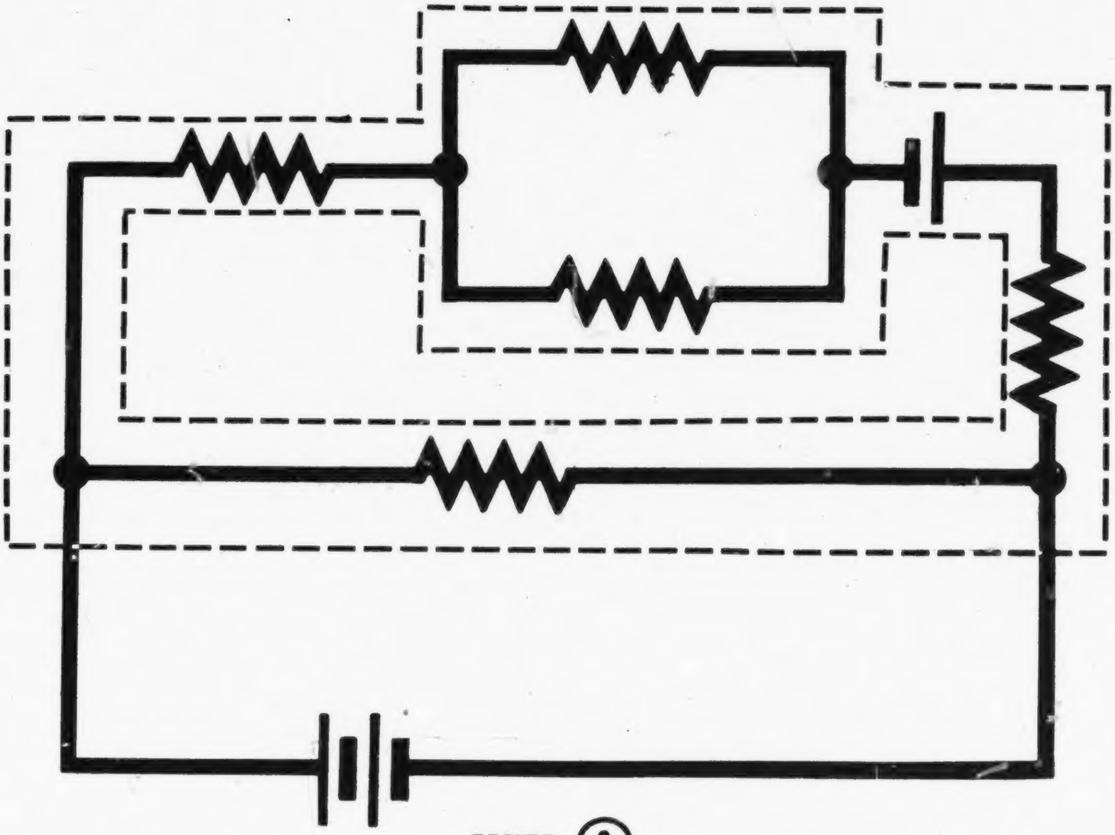


FIGURE ③

KIRCHHOFF'S RULES

1. AT ANY JUNCTION, THE ALGEBRAIC SUM OF THE CURRENTS MUST BE ZERO.

FIGURE ④

FIGURE 5

The second of Kirchhoff's Laws states that: The sum of the changes of potential encountered in making a complete loop is zero. More explicitly, one starts at any point in the loop, traverses the loop in an arbitrary direction, and algebraically sums the potential differences met in traversing the loop. Kirchhoff's Second Law requires that this sum be zero.

Now that the two Kirchhoff rules have been stated, they may be applied to the simple circuit shown in Figure 6.

KIRCHHOFF'S RULES

2. THE SUM OF THE CHANGES IN POTENTIAL ENCOUNTERED IN MAKING A COMPLETE LOOP IS ZERO.

FIGURE ⑤

FIGURE 6

In the circuit shown in the figure there are two seats of emf, E_1 = 24 volts and E_2 = 12 volts. In addition, there are three resistors of 8 Ω, 4 Ω, and 6 Ω as shown. The author advises the reader to sketch the circuit of Figure 6 so that he may later follow the solution to this problem on his own.

The following problem is presented as an illustrative example: Determine the current in each of the resistors of Figure 6.

FIGURE 7

The first step in a problem of this type is to note the number of loops in the circuit. Inspection shows that this circuit has two loops. There is a loop on the left hand side which is assumed to have a clockwise current i_1. There is also a loop on the right hand side which is assumed to have a clockwise current i_2. The next step in the solution is shown in Figure 8.

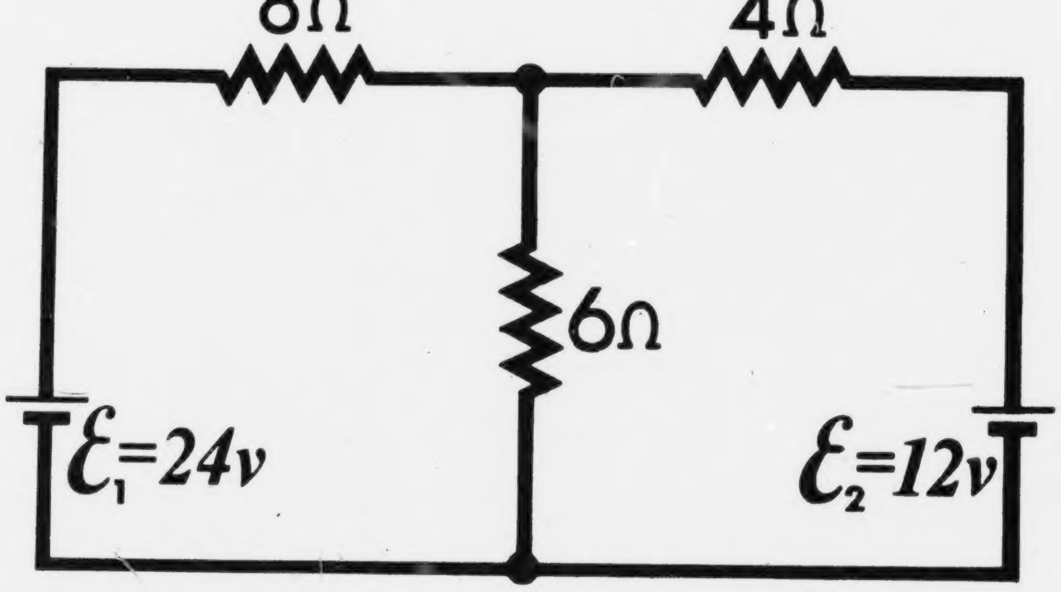

8Ω
4Ω
6Ω
$\mathcal{E}_1 = 24v$
$\mathcal{E}_2 = 12v$

FIGURE ⑥

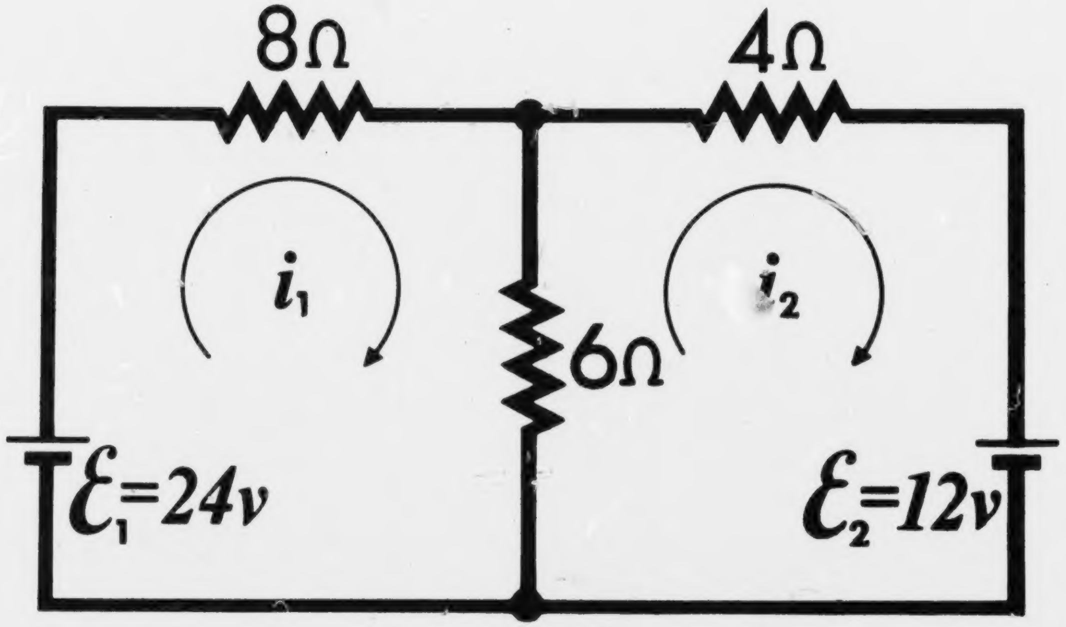

8Ω
4Ω
i_1
i_2
6Ω
$\mathcal{E}_1 = 24v$
$\mathcal{E}_2 = 12v$

FIGURE ⑦

FIGURE 8

If Kirchhoff's Second Rule is applied to the first loop, one obtains as a voltage equation

$$E_1 = i_1 (8 \Omega + 6 \Omega) - i_2 (6 \Omega)$$

Introducing the known value of E_1 (24 volts) this equation becomes

$$24 = i_1 (14 \Omega) - i_2 (6 \Omega)$$

See Figure 9.

FIGURE 9

Attention is now given to the second loop. Here Kirchhoff's Second Rule yields

$$- E_2 = i_2 (6 \Omega + 4 \Omega) - i_1 (6 \Omega)$$

Replacing $- E_2$ by the known value of 12 volts, this equation becomes

$$- 12 = i_2 (10 \Omega) - i_1 (6 \Omega)$$

At this time the reader should be asking himself why the voltage E_2 is written as minus 12 volts. The equation, however, is correct as written. Please turn to Figure 10.

$$\mathcal{E}_1 = i_1\left(8\Omega + 6\Omega\right) - i_2\,6\Omega$$

$$24v = i_1\ 14\Omega - i_2\ 6\Omega$$

FIGURE ⑧

$$-\mathcal{E}_2 = i_2\left(6\Omega + 4\Omega\right) - i_1\ 6\Omega$$

$$-12v = i_2\ 10\Omega - i_1\ 6\Omega$$

FIGURE ⑨

FIGURE 10

The problem has been reduced to a simple pair of simultaneous equations in the unknowns i_1 and i_2. If the reader wishes to solve these equations by himself he may do so and then proceed to Figure 11. If not, proceed to Figure 11 at once.

FIGURE 11

As can be seen from the figure, the results of solving the simultaneous equation are

$$i_1 = 1.62 \text{ AMPS}$$

$$i_2 = -.25 \text{ AMPS}$$

The negative value of i_2 merely indicates that the wrong direction was assumed for i_2. Accordingly, the current diagram must be modified as is shown in Figure 12.

FIGURE 12

Referring to the diagram of the circuit, i_1 gives the current in the 8 ohm resistor and i_2 gives the current in the 4 ohm resistor. Thus the current in these two resistors is determined. Please go on to Figure 13.

$$24v = i_1\ 14\Omega - i_2\ 6\Omega$$

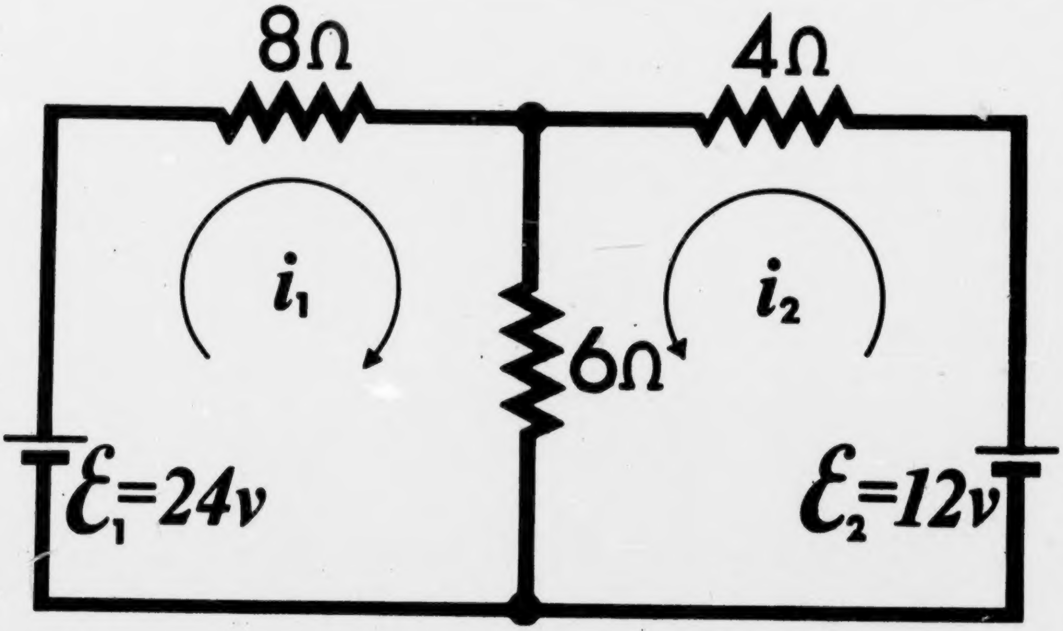
$$-12v = i_2\ 10\Omega - i_1\ 6\Omega$$

FIGURE ⑩

$$i_1 = 1.62\ amps$$

$$i_2 = -.25\ amps$$

FIGURE ⑪



8Ω
4Ω
i_1
i_2
6Ω
$\mathcal{E}_1=24v$
$\mathcal{E}_2=12v$

FIGURE ⑫

FIGURE 13

It remains for the current in the 6 ohm resistor to be determined. This can be accomplished by an application of Kirchhoff's First Rule, the rule which defines junctions. Recall that Kirchhoff's First Rule states that: the algebraic sum of the currents entering a junction and the currents leaving a junction is zero.

FIGURE 14

Figure 14 shows the corrected directions of the currents i_1 and i_2. Let the current through the 6 ohm resistor be designated by i. According to Kirchhoff's First Rule,

$$i_1 + i_2 + i = 0$$

If the currents entering the junction are taken as positive and those leaving the junction are taken as negative, this current equation becomes

$$1.62 \text{ AMPS} + .25 \text{ AMPS} - i = 0$$

or

$$i = 1.87 \text{ AMPS}$$

Thus the problem has been completed by finding the current in the 6 ohm resistor to be 1.87 AMPS.

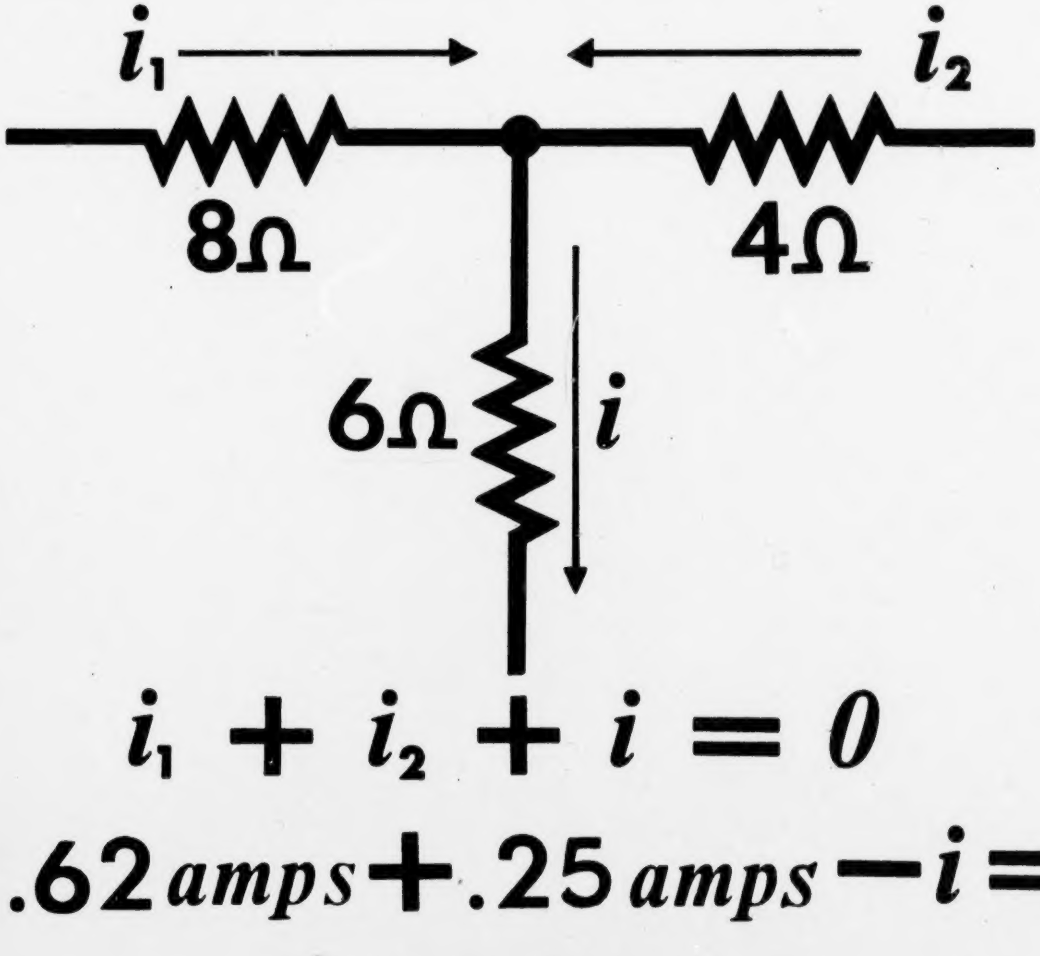
It is important to note that the two loops with which the problem was begun were rather arbitrarily chosen. The author now wishes the reader to solve the same problem using a slightly different approach. Naturally, the same result is expected. The diagram for this exercise can be found in Figure 15.

$$24v = i_1\ 14\Omega - i_2\ 6\Omega$$

$$-12v = i_2\ 10\Omega - i_1\ 6\Omega$$

Hence $i_1 + i_2$

FIGURE ⑬


i_1
i_2
8Ω
4Ω
6Ω
i

$$i_1 + i_2 + i = 0$$

$$1.62\,amps + .25\,amps - i = 0$$

$$i = 1.87\,amps$$

FIGURE ⑭

FIGURE 15

The two loops that should be used are outlined in the figure. The current through the 8 ohm and 6 ohm resistors is i_1. The current through the 8 ohm and 4 ohm resistors is i_2. It is important to note that i_2 passes through both seats of emf. This problem may be solved using the same technique as was used in the illustrative example given above. For some additional hints see Figure 16.

FIGURE 16

Using Kirchhoff's First Rule, an equation may be written for the i_1 loop.

$$- E_1 = i_1 \ (6 \ \Omega + 8 \ \Omega) - i_2 \ (8 \ \Omega)$$

Proceeding in the same way for the i_2 loop yields

$$E_1 - E_2 = (8 \ \Omega + 4 \ \Omega) - i_1 \ (8 \ \Omega)$$

Upon substituting the given values of E_1 and E_2 into these equations, there results two simultaneous equations in i_1 and i_2. When this pair of equations is solved, Kirchhoff's First Rule may be applied to yield the current in each of the resistors.

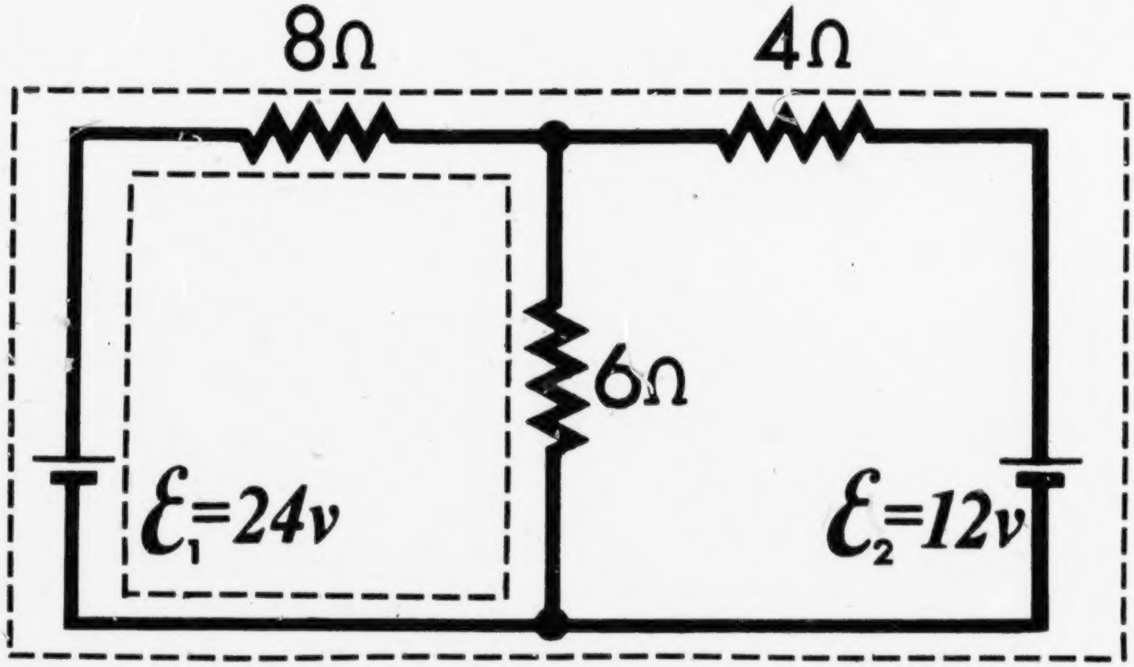

8Ω
4Ω
6Ω
ℰ₁=24v
ℰ₂=12v

FIGURE ⑮

$$-\mathcal{E}_1 = i_1\,6\Omega + i_1\,8\Omega \\ - i_2\,8\Omega$$

$$\mathcal{E}_1 - \mathcal{E}_2 = i_2\left(8\Omega + 4\Omega\right) \\ - i_1\,8\Omega$$

FIGURE ⑯

KIRCHHOFF'S RULES

TERMINAL OBJECTIVES

13/1 B Answer questions relative to the methods of application of Kirchhoff's Current Law to electrical networks.

13/1 D Apply Kirchhoff's Laws to the solution of numerical problems ranging from simple to more complex multiloop networks.

DEFINITION OF "B" FIELD

The concept of a vector field is normally first encountered in mechanics when gravitation is studied: any of the phenomena involving gravitation that are described in terms of a force acting at a distance can also be analyzed by means of the field approach, often more successfully. The same is true of the forces involved in electrostatics: although one may speak of the attraction and repulsion of electric charges as forces acting over a distance, the description can almost always be enhanced by introducing the concept of the vector field, in this case the electrostatic or electric field.

The third type of vector field is the subject of this exposition, namely the B-field or magnetic field. Since there is a strong similarity among the methods used to detect and measure all three of these vector fields, it would be profitable to review these methods as applied to gravitational and electrostatic fields before starting the analysis of the B-field.

Man is equipped by Nature to detect the presence of a gravitational field. He feels the force exerted on his body and objects he handles by the interaction of these masses with the gravitational field. As illustrated in Figure 1, he defines and measures the field with the help of a simple device such as a scale or balance and a standard mass. According to conventions of scientific mensuration, the magnitude and direction of the force of gravity acting on a one-kilogram mass provides all the information required to describe the intensity and sense of the gravitational field, the intensity is defined as nothing more than the force per unit mass.

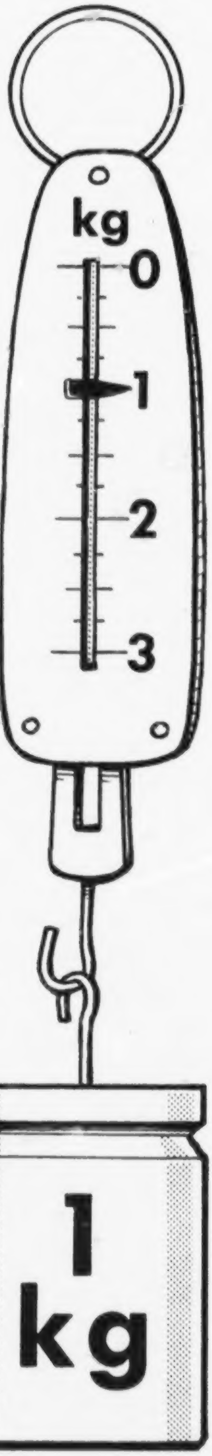


FIGURE ①

The electrostatic field presents a different problem because special instruments are required to assist with detection and measurement. Among the simplest of these is a very light, suspended object such as a pith ball and some form of electrostatic generator. (Figure 2). A pith ball is suspended from an insulating thread and given a positive charge by touching it to a glass rod that has been stroked with silk cloth. The pith ball is then brought near the electrostatic generator and the force acting on the pith ball observed by noting whether it swings toward or away from the source of the electrostatic field. A measure of the electric field around the generator is provided by the direction and magnitude of the force on the ball. The intensity of the electric field is then defined as the force per unit charge. Thus, in both of these cases, we describe the field in terms of the force acting on a unit "something" -- a unit mass in one case and a unit charge in the other.

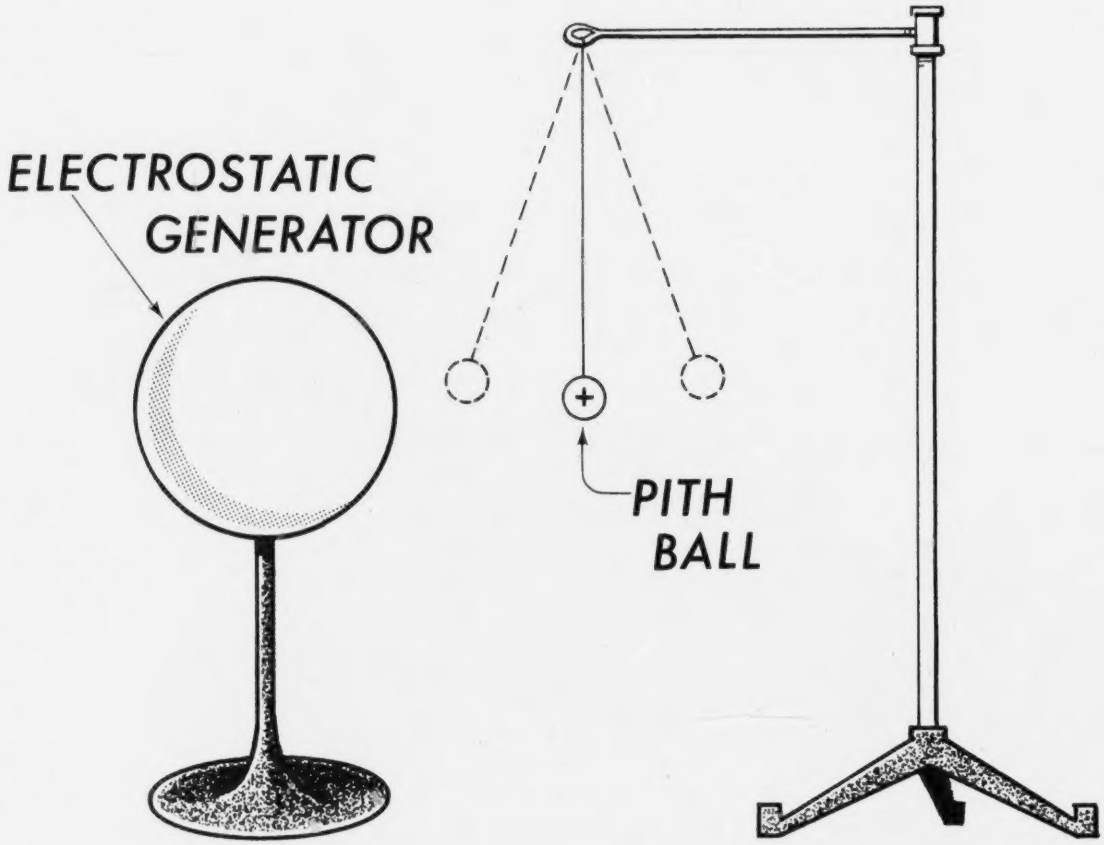

ELECTROSTATIC GENERATOR
+
PITH BALL

FIGURE ②

As might be anticipated, detection and measurement of the B-field presents problems of its own. If, as described in Figure 3, a positively charged pith ball is suspended in the magnetic field produced by a very powerful magnet, no force appears to act on either the magnet or the pith ball. If the pith ball is swept rapidly through the magnetic field, however, a force does make itself evident: the pith ball is observed to be deflected sidewise with respect to the direction of its motion through the field. The most noticeable affect is obtained when the ball passes between the poles of the magnet moving at right angles to the axis joining the pole faces. This is illustrated in Figure 4. The pole axis is a straight line (shown dotted) joining the centers of the two flat pole faces: the pith ball is suspended immediately below the pole axis with the thread intersecting the axis as shown. When the magnet is moved quickly downward causing the pith ball to pass perpendicularly through the field between poles, the pith ball is seen to deflect horizontally toward the observer.

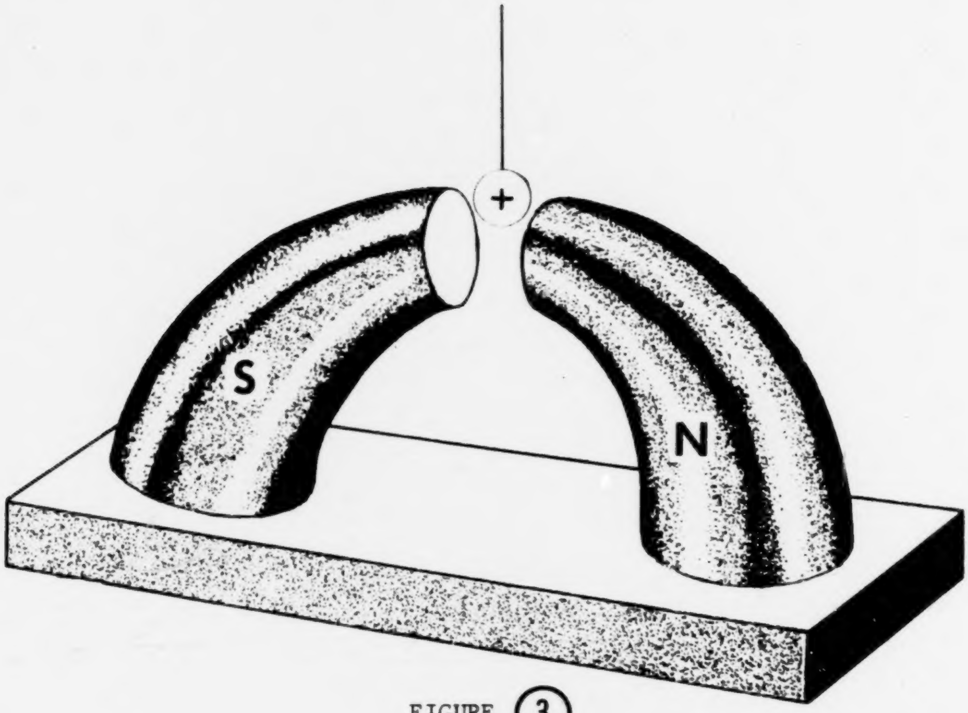


FIGURE ③

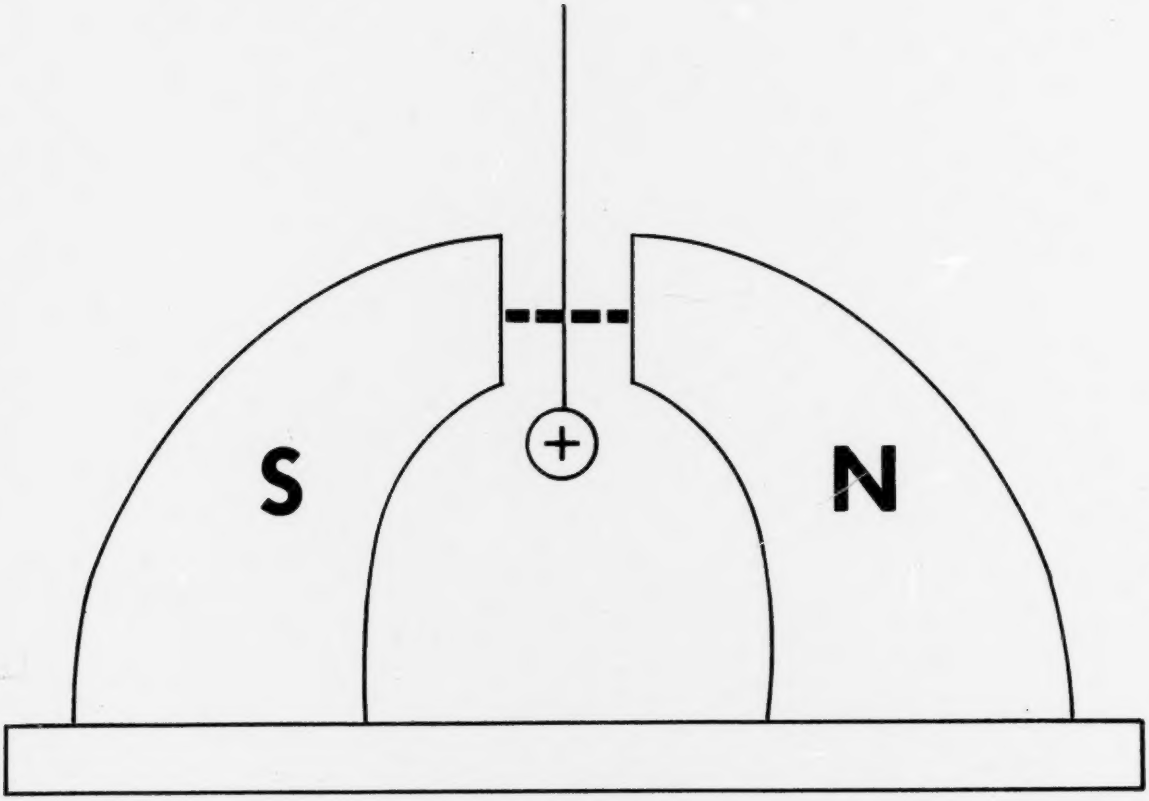


FIGURE ④

The field of the magnet in Figure 4 is strongly concentrated between the poles and is arbitrarily assigned the direction from the N-pole toward the S-pole within the gap between them. Please refer to Figure 5. The field direction is shown as the vector arrow $\vec{B}$ from right to left, that is, from N to S. The relative velocity of the pith ball in the field is indicated by the vector arrow $\vec{v}$, and the force resulting from the motion of the charged body through the magnetic field is shown by the vector arrow $\vec{F}$.

From a purely descriptive point of view, it is important to observe that the force $\vec{F}$ is perpendicular to the plane containing vectors $\vec{v}$ and $\vec{B}$. In this case, $\vec{F}$ is directed toward the observer but if either $\vec{v}$ or $\vec{B}$ had been oppositely directed, the sense of $\vec{F}$ would be away from the observer but it would still be perpendicular to the v-B plane.

Analytically, it should be apparent that the force $\vec{F}$ is related to $\vec{v}$ and $\vec{B}$ by the cross-product of these terms. If the upward velocity $\vec{v}$ is rotated into $\vec{B}$, and if the resulting motion of a right-handed screw is visualized, it is at once seen that the screw would progress at right angles to both $\vec{v}$ and $\vec{B}$ out of the paper toward the observer.

Thus far, then, it is seen that a force does act on a charge in a magnetic field BUT ONLY IF THE CHARGE IS MOVING WITH RESPECT TO THE FIELD. It can be demonstrated furthermore that this force will exist only if the relative action of the pith ball with respect to the field has a component perpendicular to the field. When the charge moves parallel to the field, say along the pole axis, no force can be detected. In any case -- if the force can be detected -- it is always found to be perpendicular to the plane containing the $\vec{v}$ vector and the $\vec{B}$ vector.

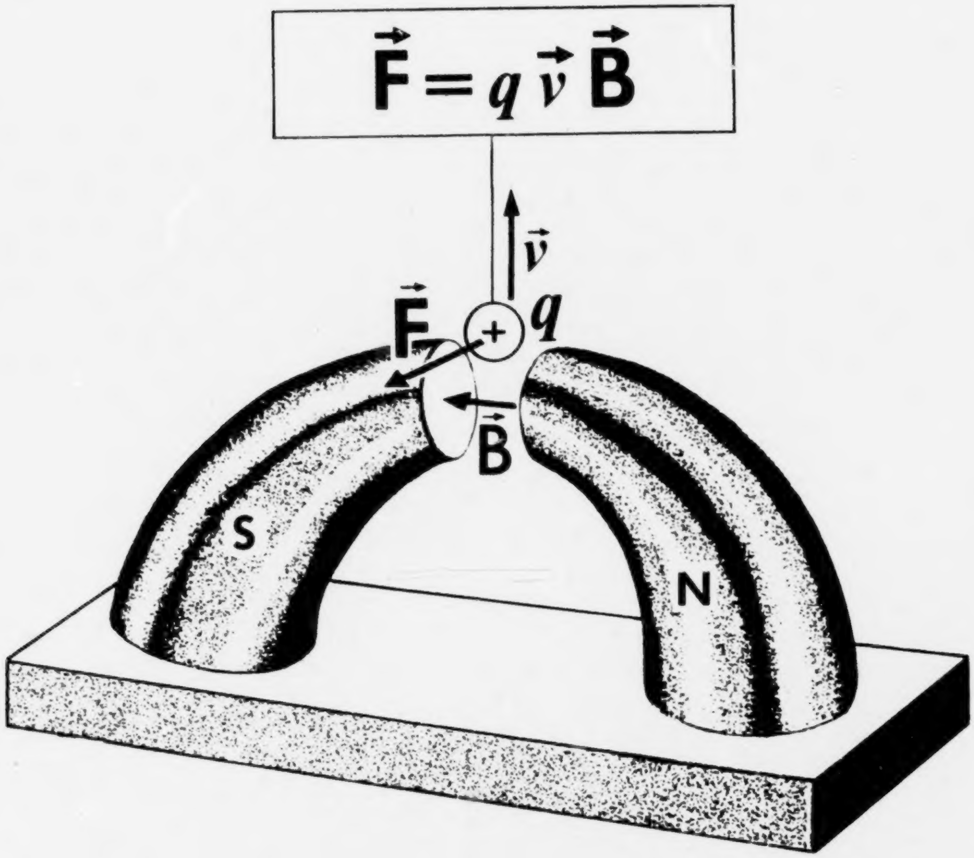

$\vec{F} = q\vec{v}\vec{B}$
$\vec{v}$
$\vec{F}$
+
q
$\vec{B}$
S
N

FIGURE ⑤

Another revealing experimental set up is given in Figure 6. An evacuated glass bulb containing two electrodes, called a Crooke's Tube, is connected to a source of high voltage. When power is applied, a stream of electrons made visible by a fluorescent screen in the tube (not shown in the diagram) passes in a straight line from the negative to the positive electrode. The electron stream is indicated by the broken line inside the Crooke's Tube. If a magnet is placed behind the tube from the observer's point of view, horizontally oriented so that the N-pole of the magnet points toward the observer, the electron stream is seen to be deflected sharply upward. The deflected path is represented by the dotted arrow in the figure.

The electrons are moving from electrode A to electrode C, from left to right in this case: the direction of the B-field is perpendicular to the plane of the diagram and its sense is outward toward the observer away from the N-pole behind the tube. Thus, the B-field is perpendicular to the velocity vector of the electron stream. The deflection of the stream provides evidence that a force is exerted on each electron in an upward direction, perpendicular to the plane containing the $\vec{v}$- and $\vec{B}$- vectors. This result is in descriptive agreement with the observations of the previous experiment.

There is a significant difference, however, between the two demonstrations: in the first, a positive pith ball moved with respect to the B-field but in the second the moving charges were negative. When one tries to rotate $\vec{v}$ into $\vec{B}$ in this case, one finds that the sense of the predicted force should be downward rather than upward. Evidently, since the various vector rotation rules and the rules governing the directions of forces on charged particles in fields are based on the motion of POSITIVE CHARGES, it is necessary to revise the approach to the problem when negative charges are involved. This is rather easily done as follows:

Theory and experiment demonstrate that an electron moving from left to right as in our example has exactly the same field effect as a positive charge of the same magnitude and mass moving in the opposite direction, from right to left, Thus, to correct the vector picture when dealing with negative charges it is helpful to redraw the diagram as shown in Figure 7. Nothing has been changed except the direction of the charges, these having been changed to positrons instead of electrons.

CROOKE'S TUBE

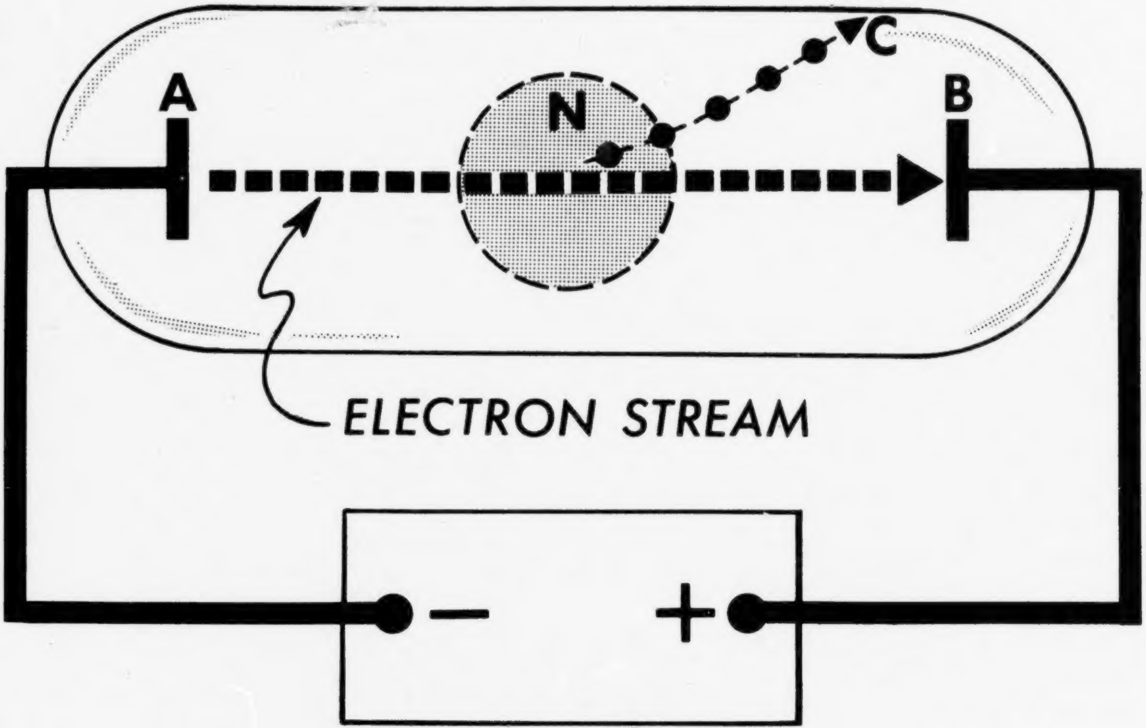

C
A
N
B
ELECTRON STREAM
–
+

FIGURE ⑥

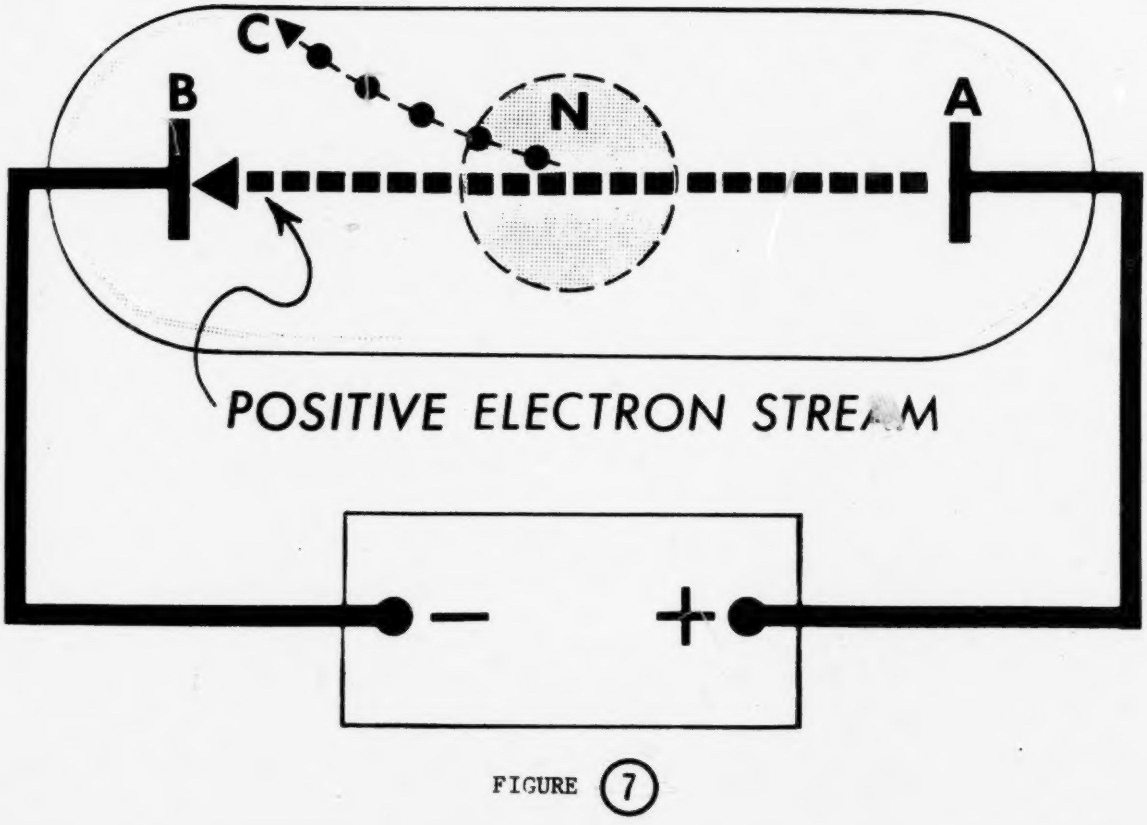

C
B
N
A
POSITIVE ELECTRON STREAM
–
+

FIGURE ⑦

Now the rule for cross-products may be applied without error. When $\vec{v}$ is rotated into $\vec{B}$, the direction of progress of the right-handed screw is upward as it should be. This is diagrammed in Figure 8. The velocity vector $\vec{v}$ is directed toward the left, the field vector $\vec{B}$ is directed outward toward the observer, and the resulting force $\vec{F}$ is upward.

The force acting on the positively charged particles, as indicated in the figure, has a magnitude given by the product $q\,\vec{v} \times \vec{B}$ in which the velocity vector $\vec{v}$ is perpendicular to the field vector $\vec{B}$.

The units for B are readily obtained from the expression

$$F = q\,v\,B$$

by solving this equation for B, that is

$$B = \frac{F}{q\,v}$$

and then substituting mks units in the term at the right:

$$B = \frac{\text{newtons}}{\text{coulombs}\cdot\frac{\text{meter}}{\text{second}}}$$

A coulomb per second is called an ampere, hence

$$B = \frac{\text{newtons}}{\text{ampere}\cdot\text{meter}}$$

The quantity B is variously called the intensity of a magnetic field, magnetic induction, and flux density. If this material has not already been introduced, it will later be shown that another unit connected with the concept of flux density is frequently used. That is,

$$B = \frac{\text{webers}}{\text{square meter}}$$

And finally that the weber per square meter is now called the tesla. These four units are completely equivalent and one may be substituted for the other at will. For record purposes, these units are summarized in Figure 9.

$$\boxed{\vec{F} = q\,\vec{v} \times \vec{B}}$$

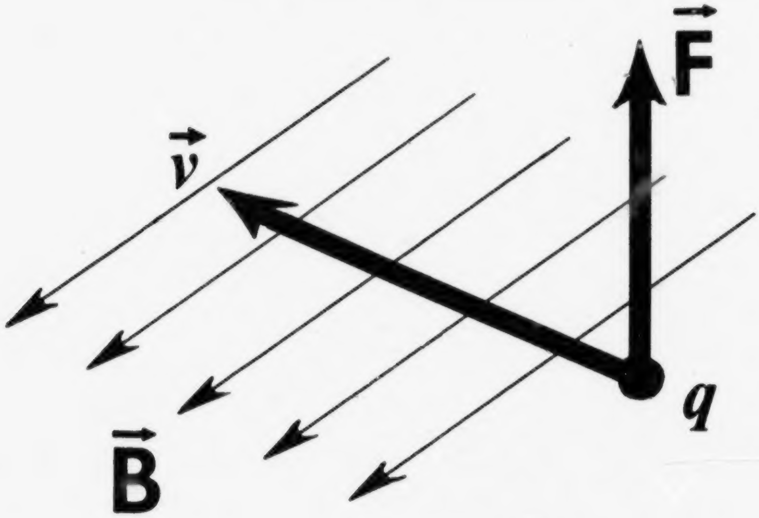

$\vec{F}$
$\vec{v}$
q
$\vec{B}$

FIGURE ⑧

UNITS of "B"
(a vector quantity)

$$\frac{nt}{coul\,(m/s)} = \frac{nt}{amp\ m}$$

$$\frac{weber}{m^2} = tesla$$

FIGURE ⑨

DEFINITION OF "B" FIELD

TERMINAL OBJECTIVES

14/1 B Answer qualitative questions relating to the magnetic induction vector B.

FORCE BETWEEN PARALLEL CURRENT-CARRYING CONDUCTORS

If two wires are freely suspended very close to one another, and if a current is then passed through each of the wires, a force of attraction or repulsion can be detected between them. The direction of the force is a function of the relative current directions; if the current directions are the same in each wire, the conductors will attract one another but if the direction of the current in one of the wires is reversed, the force changes to repulsion. Please refer to Figure 1.

Analysis of the electromagnetic fields that surround each conductor indicates that both the magnitude and the direction of the force can be theoretically predicted. Let us assume that the wires shown in Figure 2 are connected directly to a source of emf, in series with one another, so that the currents are opposite in direction but equal in magnitude.

The current in wire a is directed downward while that in wire b is upward. In order to make the analysis easier to perform in two dimensions, imagine that both wires have been rotated about a horizontal axis so that they present the picture shown in Figure 3. The wires now appear in cross-section as small discs; wire a carries a dot to indicate that the current is directed toward the observer and wire b contains a cross to show that the current in this wire is directed into the plane of the paper, away from the observer. Considering wire a alone for the moment, as in Figure 4, the B-lines surrounding it may be drawn as concentric circles to conform with experimental facts obtained from Oersted's Experiment.

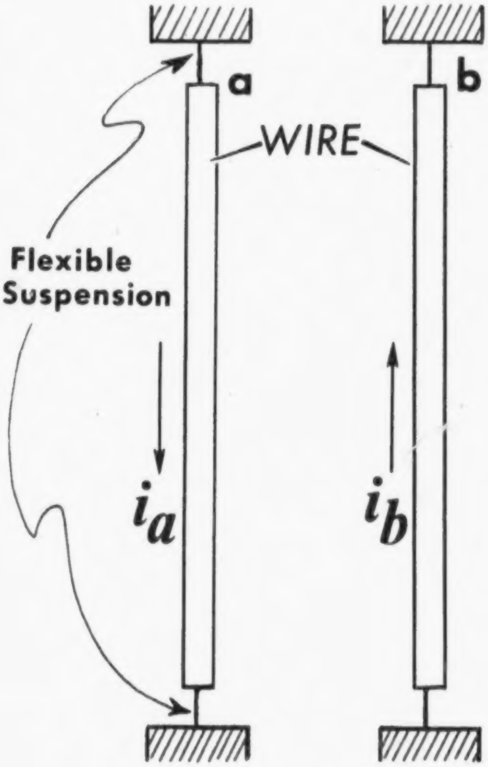

a
b
WIRE
Flexible Suspension
i_a
i_b

FIGURE ①

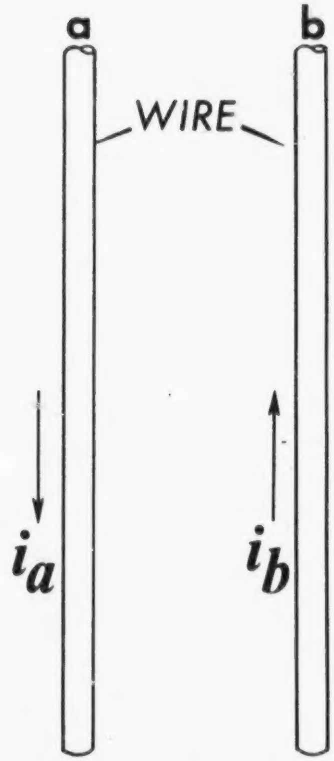

a
b
WIRE
i_a
i_b

FIGURE ②

FIGURE ③

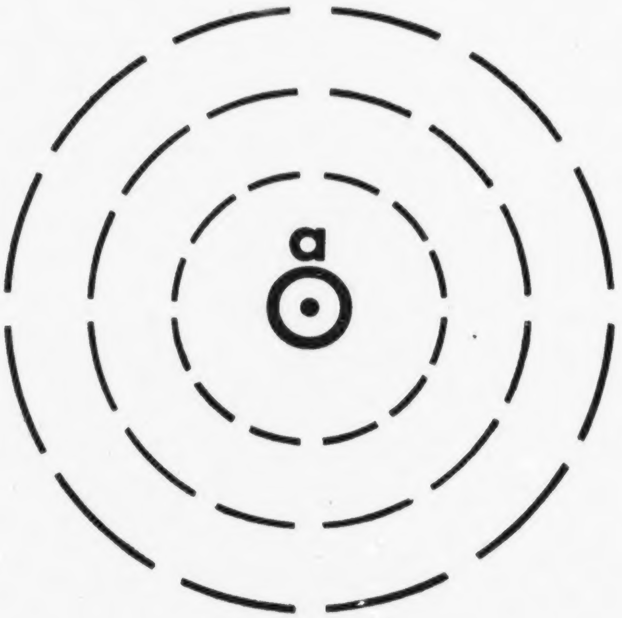


FIGURE ④

Applying the right-hand rule for wires (Oersted's Rule), the thumb of the right hand is pointed in the direction of the conventional current so that the fingers then encircle the wire in the direction of the magnetic field. For this case, the B-lines are counterclockwise in direction as indicated in Figure 5. At a point P near the current-carrying wire, the line of magnetic induction is tangent to the circle of the B-line surrounding the wire.

The magnitude of the field at point P is given by Ampere's Law and may be written as indicated in Figure 6, in which B_p is the magnitude of the field, u_o is the permeability constant, i_a is the current in wire a, and r is the distance between the center of wire a and point P.

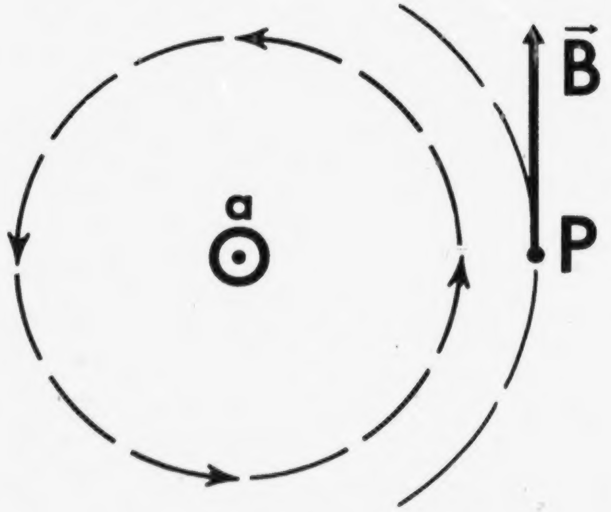
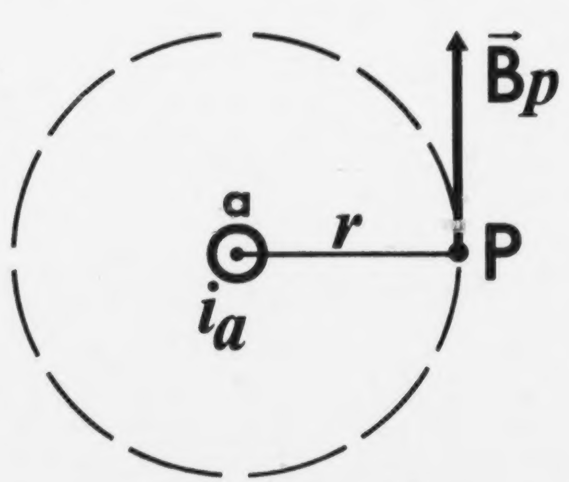


FIGURE ⑤



$$B_p = \frac{\mu_o\, i_a}{2\pi r}$$

FIGURE ⑥

To review another concept briefly, please refer to Figure 7. In this diagram, a wire is immersed in a magnetic field; the wire carries a current into the plane of the diagram. The source of the magnetic field is not indicated, nor is this information needed to analyze the problem. The B-lines from this unknown source are directed upward in the plane of the paper as indicated. Applying the Palm Rule to determine the direction of the force acting on the current-carrying conductor immersed in the given field, the fingers of the right hand are placed so that they point in the direction of the B-lines while the extended thumb points in the direction of the current. The direction of the force on the wire is then given by the direction in which the palm would exert a thrust if the hand were used in the normal manner. In the example given in Figure 5, the direction of the force would be that shown in Figure 8, namely to the right as viewed by the observer.

The Palm Rule may always be used in this way and will be found to be a great help in analyzing this kind of situation and others similar to it.

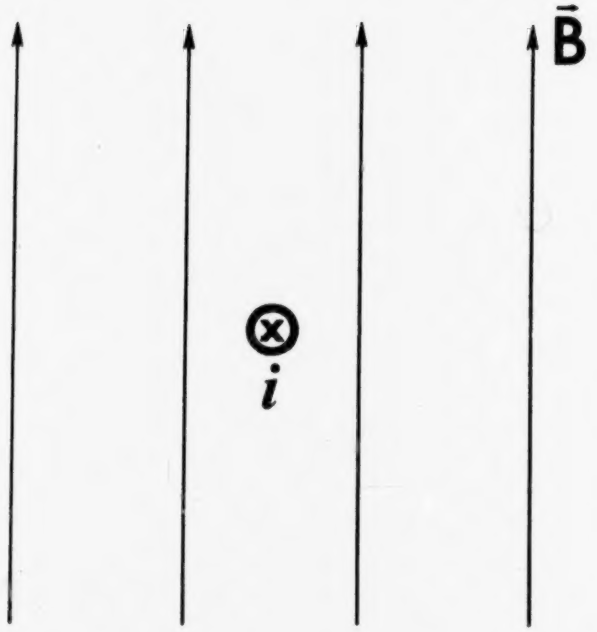


FIGURE ⑦

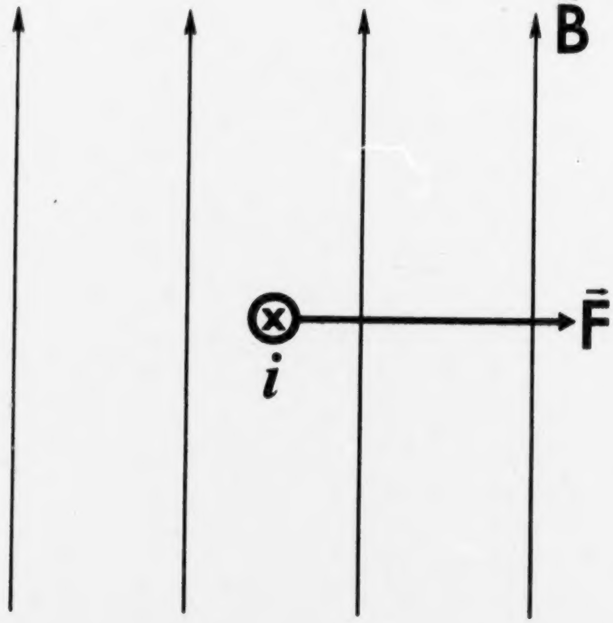


FIGURE ⑧

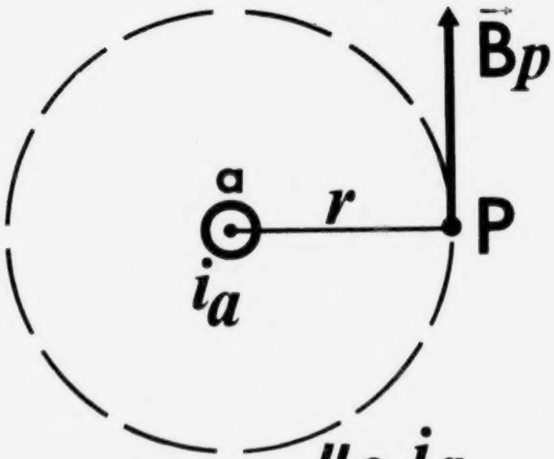
The magnitude of the force on the current-carrying wire is given by the relation shown in Figure 9. Thus, both the magnitude of the force and its direction are determinable for the example given. Please refer to Figure 10; this is reiteration for review. Also refer to Figure 11.

These ideas may now be combined to determine the nature of the force in a specific case; that is, to determine whether to expect attraction or repulsion when the current directions are known. Working with conductors carrying oppositely directed currents as in Figure 12, it can be readily shown that the force is one of repulsion in the following manner.

The line of magnetic induction at wire b due to the current in wire a is labeled B_a. Applying the Palm Rule to wire b, it is seen that the force on this wire is directed to the right away from wire a as illustrated in Figure 13. The magnitude of the force is given in the same Figure. In this relationship, F_b is the force acting on wire b, i_b is the current in wire b, l_b is the length of wire b, and B_a is the magnetic induction due to the current in wire a.

$$F = il\ B_{\perp}$$

FIGURE ⑨


$\vec{B}p$
a
r
P
i_a

$$Bp = \frac{\mu_0\, i_a}{2\pi r}$$

FIGURE ⑩

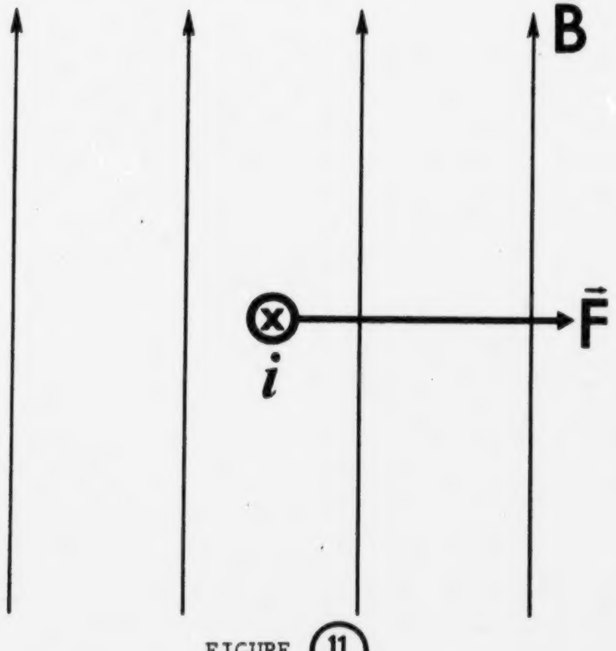


FIGURE ⑪

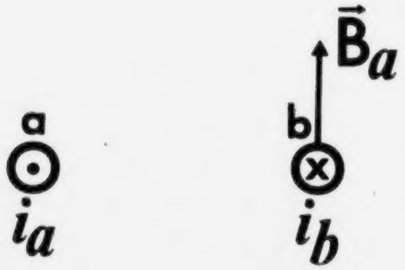
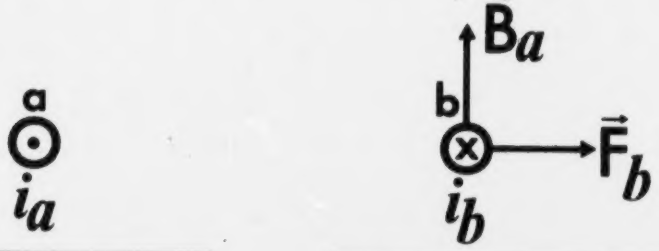

$\vec{B}_a$
a
b
i_a
i_b

FIGURE ⑫

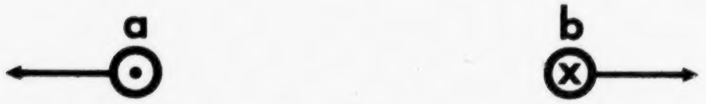

$\vec{B}_a$
a
b
$\vec{F}_b$
i_a
i_b

$$\boxed{F_b = i_b\ l_b\ B_a}$$

FIGURE ⑬

Exactly the same process may be followed to find the force acting on wire a due to the current in wire a and the magnetic induction produced by the current in wire b. The right-hand rule is first applied to wire b; this demonstrates that the B-line at wire a is directed upward. Then the Palm Rule is applied to wire a, showing that the force on this wire acts to the left away from wire b. The direction and magnitude of this force is diagrammed in Figure 14. The student should confirm this for himself.

Thus, the wires repel each other. From Third Law considerations alone, one may conclude that the force on wire a must equal the force on wire b since they form an action-reaction pair. The fact that the forces are equal may also be shown directly as in Figure 15. In the first step, the magnitude of F_b is given in equation form. In the second step, B_a has been replaced by its equivalent, i.e., $\mu_o i_a/2\pi r$. Both sides are then divided by the wire length to yield the force per unit length in the third step. The remainder is self-explanatory.



$$\boxed{F_a = i_a\, l_a\, B_b} \qquad \boxed{F_b = i_b\, l_b\, B_a}$$

FIGURE ⑭

$$F_b = i_b\, l_b \underbrace{\frac{\mu_o\, i_a}{2\pi r}}_{B_a}$$

$$\frac{F_b}{l_b} = \frac{\mu_o\, i_b\, i_a}{2\pi r}$$

and assuming equal
lengths and currents

$$\frac{F}{l} = \frac{\mu_o\, i^2}{2\pi r} \text{ for either wire}$$

FIGURE ⑮

In summary, as presented in Figure 16, the force between current-carrying wires is one of REPULSION if the currents are OPPOSITELY DIRECTED; the force is ATTRACTION if the currents have the SAME DIRECTION. The force per unit length on either wire for equal currents and equal lengths is given by

$$F/l = \mu_0 i^2/2\pi r$$

Summary

REPULSION, if currents are oppositely directed;

ATTRACTION, if currents have same direction.

$$\frac{F}{l} = \frac{\mu_0 \; i^2}{2\pi r}$$

for either wire if currents are equal.

FIGURE ⑯

FORCE BETWEEN PARALLEL CURRENT-CARRYING CONDUCTORS

TERMINAL OBJECTIVES

14/3 A Describe the magnetic field around a straight-current-carrying conductor.

14/3 D Prove that the force between wires a and b in the diagram is an attractive force, the magnitude of the force on either wire being given by (equation).

AMPERE'S LAW APPLIED TO A LONG STRAIGHT CONDUCTOR

Like many other great generalizations in physics, Ampere's Law begins to acquire meaning only when it is related to phenomena that occur in the physical world. Using the accepted symbolism of physics, Ampere's Law may be stated in the form illustrated in Figure 1.

Put into words, one may read this as: "The line integral of the magnetic induction B around any closed path in a magnetic field is equal to the net current across the area, multiplied by a constant of proportionality, μ_o".

Unfortunately, the verbal expression of Ampere's Law may be just as obscure to many readers as the mathematical statement. It can be clarified to a great extent, however, by considering a specific example in which the quantities contained in Ampere's Law can be reasonably and intelligently included.

Referring to Figure 2, imagine five conductors passing through the plane of the diagram perpendicularly in more or less random positions. The wires appear as small discs carrying either a cross or a dot to indicate current either into the plane of the paper or out of it, respectively.

Figure 3 shows the five conductors enclosed in a continuous "path" which is to serve as the path for the line integral.

In the next step (Figure 4) a randomly chosen point, P, has been inserted in the closed path. The two vector arrows originating at P are, respectively, the magnetic induction vector $\vec{B}$ pointing in any random direction and an element of path length $d\vec{l}$ that is tangent to the curve of the path at point P. Since the conductors passing through the area circumscribed by the closed path must produce a magnetic field in the plane of the diagram, then the $\vec{B}$ vector must have a specific magnitude and a specific direction, the latter designated by the angle between it and the $d\vec{l}$ vector, that is, angle θ. The net current threading through the enclosed area is merely the algebraic sum of the five individual currents.

AMPERE'S LAW

$$\oint \vec{\mathbf{B}} \cdot d\vec{l} = \mu_0 i$$

FIGURE ①

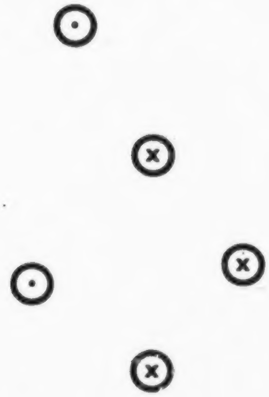


FIGURE ②

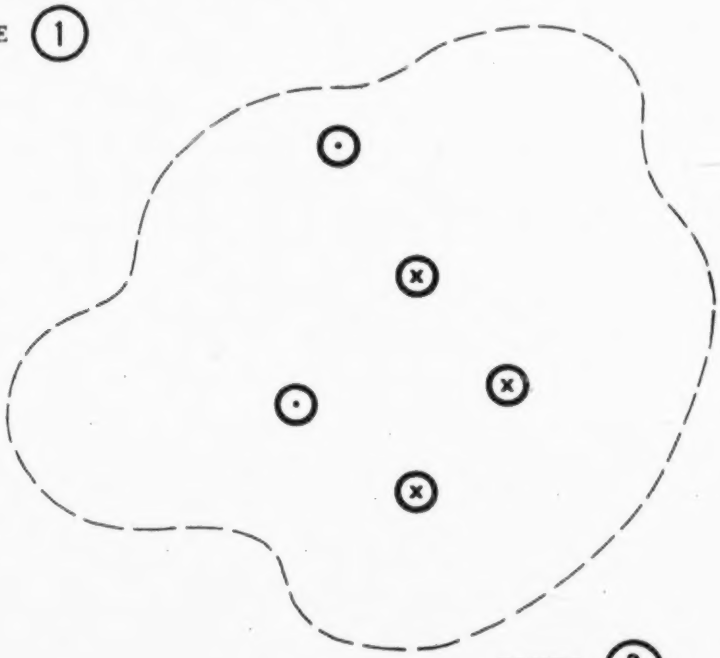


FIGURE ③

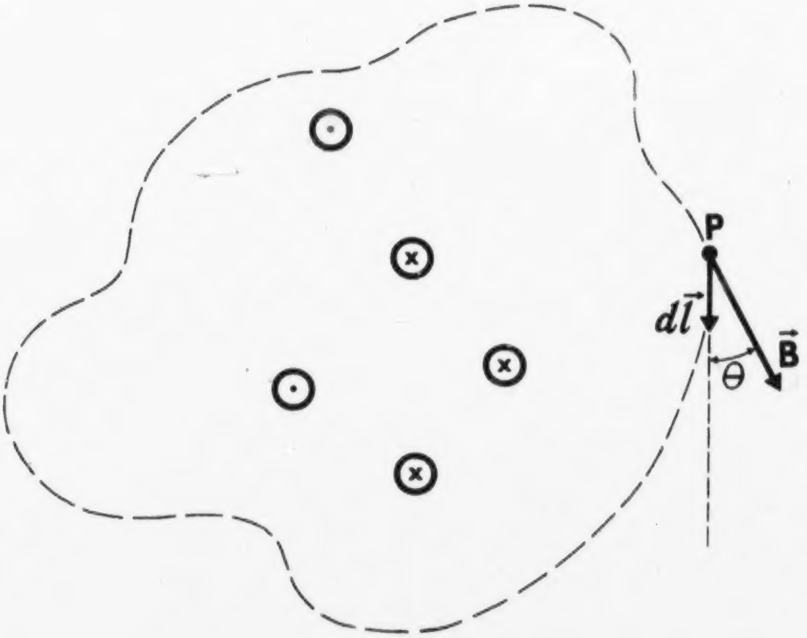

P
$d\vec{l}$
$\vec{B}$
θ

FIGURE ④

In accordance with Ampere's Law as stated in Figure 1, the sum or integral of all the $\vec{B}.d\vec{l}$ contributions over the closed path must be equal to the product of the net current as described above and the proportionality constant, μ_o. It must be emphasized at this point that Ampere's Law merely describes a general property of magnetic fields as related to the currents that produce them. It is not an "engineering formula" in which one plugs numbers in order to extract an answer; in a sense, it describes Nature but does not tell how to handle her, except in special, simple cases.

Now refer to Figure 5. Here is Ampere's Law once again, stated in its most general form. Note that μ_o is assigned a value of 4π (10^{-7}) webers per ampere meter. This value matches this constant to the mks system of units; the name given to μ_o is "permeability constant". Figure 5 also contains another item of importance: since $\vec{B}.d\vec{l}$ is a dot product, the magnitude of the $\vec{B}.d\vec{l}$ vector at any point in the closed path is the product of the path element dl and the component of the $\vec{B}$-vector parallel to the element. That is, the magnitude of the dot product is $B\ dl \cos\theta$.

Evaluation of the line integral of $\vec{B}.d\vec{l}$ is extremely difficult mathematically except in cases of high symmetry: you may remember that this is also true of Gauss' Law.

$$\oint \vec{\mathbf{B}} \cdot d\vec{l} = \mu_0 i$$

where $\mu_0 = 4\pi \, (10^{-7})$

$$web / amp \cdot m$$

and

$$\vec{\mathbf{B}} \cdot d\vec{l} = B\, d\vec{l} \, \cos\Theta$$

FIGURE ⑤

A typical case of high symmetry to which Ampere's Law may be directly applied is that of an infinitely long, straight, current-carrying conductor. A wire that is long compared to its diameter and for which the value of B is desired not too far from the wire and not too close to its ends approximates the ideal conductor sufficiently closely. Such a wire is shown in Figure 6 & 7.

The magnetic field around a conductor with these characteristics is symmetrical and may be described as comprising concentric circles in a plane at right angles to the wire. Symmetry also tells us that the magnitude of B is constant at all points on a given B-circle and that the $\vec{B}$ vector is tangent to the circle wherever we choose it. Furthermore, the angle between the line element $d\vec{l}$ and the $\vec{B}$ vector is always zero since $d\vec{l}$ is also tangent to the circle at the selected point being superimposed on the $\vec{B}$ vector as shown.

Using the circle shown in the diagram as the path of integration, Ampere's Law may then be written in vector form as given in Figure 8. When translated into scalar form it takes the form shown in Figure 9 (a).

As mentioned previously, in this simple case dl and B lie along the same straight line, that is $\theta = 0$, so that $\cos \theta = 1$ and the statement may then be written as in Figure 9(b). Also, since B is constant over the whole closed path of integration, then the law may be further simplified as in Figure 9(c). Finally, the line integral for a circle is simply the circumference of the circle or $2\pi r$ so that the line integral of $B \cdot dl$ turns out to be nothing more than $B(2\pi r) = \mu_o i$ as in Figure 9(d). Clearly, then, the magnitude of the magnetic vector any any point on the circular line of induction with radius r and a net current i across the area enclosed by the line is $\mu_o i/2\pi r$.

The mks unit breakdown for this example is illustrated in Figure 10. The student should look this over carefully to be certain that he can understand the unit relationships.

Thus, for the simple case of a long, straight, current-carrying conductor Ampere's Law gives a formula for determining the magnitude of the magnetic induction at any point near the wire and not too close to its ends in terms of the current in amperes and the distance of the point from the wire in meters.

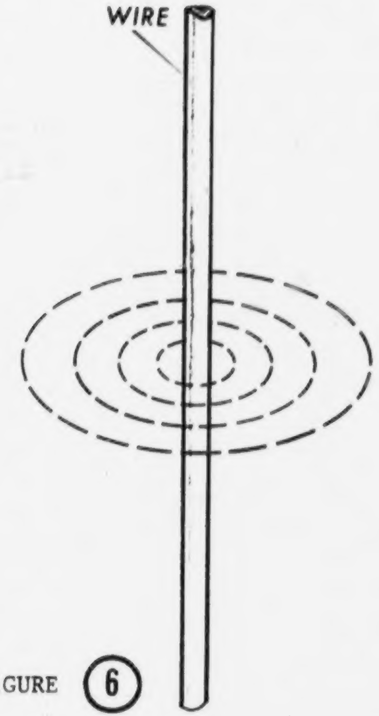


FIGURE ⑥

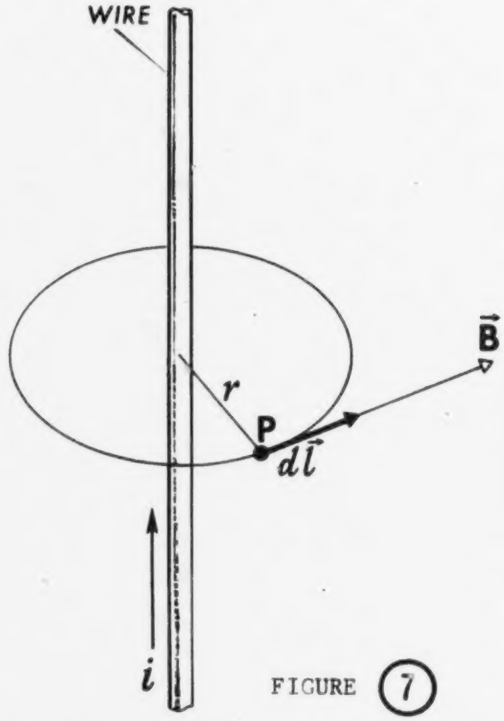


FIGURE ⑦

AMPERE'S LAW

$$\oint \vec{\mathbf{B}} \cdot d\vec{l} = \mu_0 i$$

FIGURE ⑧

FIGURE (9a) $\oint \vec{\mathbf{B}} \cdot d\vec{l} = \oint B\, dl \cos \Theta$

" (9b) $\oint B\, dl \cos \Theta = \oint B\, dl$

" (9c) $\oint B\, dl = B \oint dl$

" (9d) $B\left(2\pi r\right) = \mu_0 i$ or $B = \dfrac{\mu_0 i}{2\pi r}$

FIGURE (10) $B\left(\frac{web}{m^2}\right) = \dfrac{2i\,(amp)}{r\ (m)} \times 10^{-7} \dfrac{web}{amp \cdot m}$

AMPERE'S LAW APPLIED TO A LONG STRAIGHT CONDUCTOR

TERMINAL OBJECTIVES

14/3 A Describe the magnetic field around a straight-current- carrying conductor.

14/3 F Answer questions and solve problems involving Ampere's law and its applications.

THE LAW OF BIOT-SAVART

In a previous discussion a great deal of emphasis was placed on the fact that Ampere's Law as shown in Figure 1 is an important generalization that relates magnetic induction to the electric current that produces it. More than this, it was emphasized that Ampere's Law may be readily applied to configurations of high symmetry but that, in most cases, evaluation of the line integral is very difficult.

In such instances -- where the conditions of symmetry are not met to the extent required for applying Ampere's Law -- it is often possible to find the value of the magnetic induction vector at a point near the conductor by using a relationship called the Biot-Savart Law. Although the Biot-Savart Law may be deduced from Ampere's Law and vice versa, the proof of this is not of immediate concern at this time.

AMPERE'S LAW

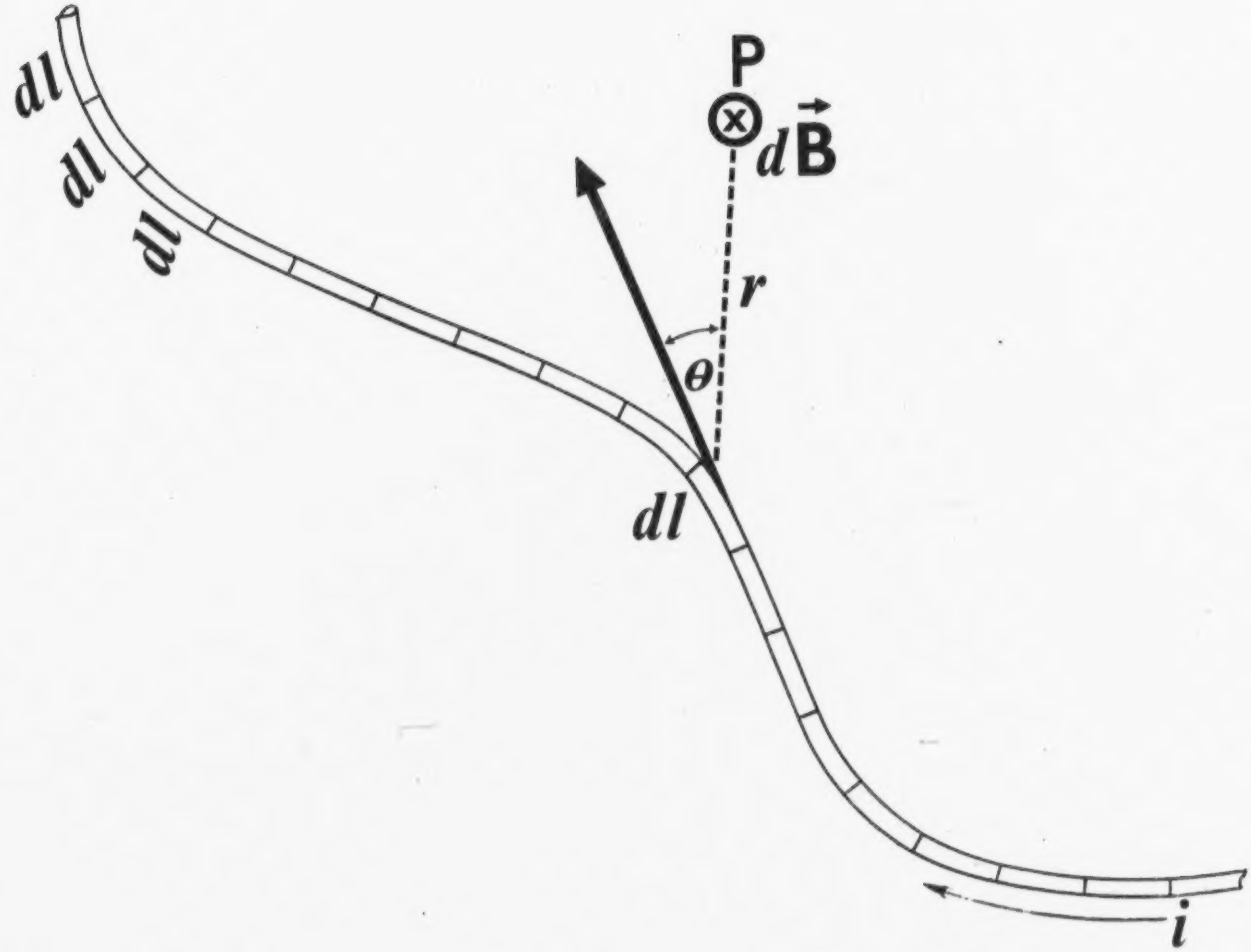
$$\oint \vec{\mathbf{B}} \cdot d\vec{l} = \mu_o i$$

FIGURE ①

This discussion will concern itself with a general statement and exposition of the Biot-Savart Law, and a description of the procedure involved in using the law to solve a specific problem.

Please refer now to Figure 2. An asymmetrically shaped wire carrying a current is shown divided into tiny elements labeled "dl". These will be referred to as "current elements". In this way, the conductor's total current is considered to be composed of a large number of discrete elements, the direction of the current in a particular element being that of the wire at that point.

If a specific current element is selected for study, one may then consider the nature of the element of magnetic induction $d\vec{B}$ that is produced by that current element. With dl and P in the plane of the diagram, the direction of the induction vector is known at point P; as given by the right-hand rule for conductors, the induction vector $d\vec{B}$ is directed into the plane of the diagram, perpendicular to it.



P
$d\vec{B}$
dl
dl
dl
r
θ
dl
i

FIGURE ②

The Biot-Savart Law is used to determine the magnitude of dB.

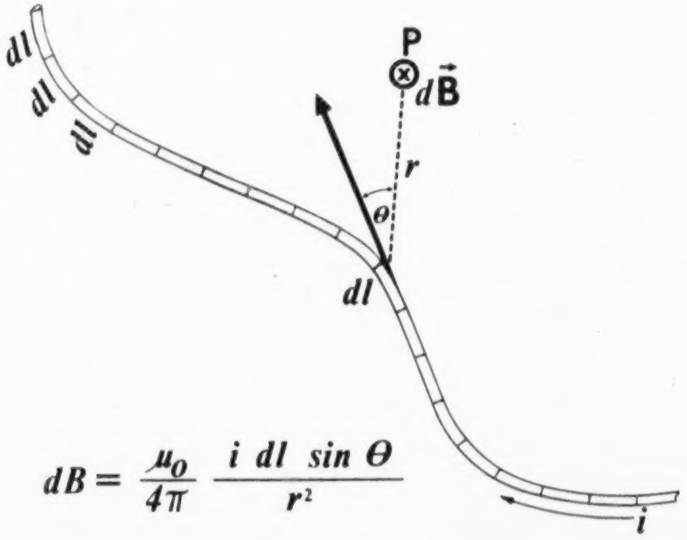
Figure 3 presents the Biot-Savart Law in mathematical form. The magnitude of the magnetic induction vector is $d\vec{B}$ at point P; the permeability constant, i is the current in the wire, dl is the length of the current element, r is the distance from the current element to point P, and θ is the angle between dl and r.

The next consideration follows logically: to find the magnitude of the induction vector at point P due to the net effect of all the dl contributions, it will be necessary to perform an integration of these elements over the whole length of the conductor.

With a constant current in a wire of specific length, the integration can be successfully performed if the variation of θ with respect to each of the current elements can be expressed mathematically over the length of the wire.

The use of the law may be readily demonstrated for a specific example. Please refer to Figure 5 which shows a long, straight, current-carrying wire for which the magnetic induction B at point P near the wire is to be determined. This example has been chosen so that the student may have the opportunity to compare the Biot-Savart solution with that obtained by using Ampere's Law in a previous case. Most textbooks discuss this particular Biot-Savart application and the student is asked to study the solution given in the books carefully. The approach used here is somewhat different, however, and provides an opportunity to see how the problem may be approached from a different starting point.

Consider the wire in Figure 5 to be infinitely long. As shown in the diagram, θ is the angle between dl and r while angle α is its complement; R is the perpendicular distance between the wire and the point P. Also, dα is the angle subtended by the length of one of the current elements. Since the wire is infinitely long, consideration of one dl after another starting at minus infinity and going up to plus infinity will involve letting α vary from -90 degrees to +90 degrees. Thus, the limits of integration extend from $-\pi/2$ to $+\pi/2$.


dl
dl
dl
P
$d\vec{B}$
r
θ
dl
i

$$dB = \frac{\mu_0}{4\pi}\,\frac{i\,dl\,\sin\Theta}{r^2}$$

FIGURE ③

$$\vec{B}_p = \int d\vec{B}$$

FIGURE ④

$$= \int \frac{\mu_0}{4\pi}\,\frac{i\,dl\,\sin\Theta}{r^2}$$

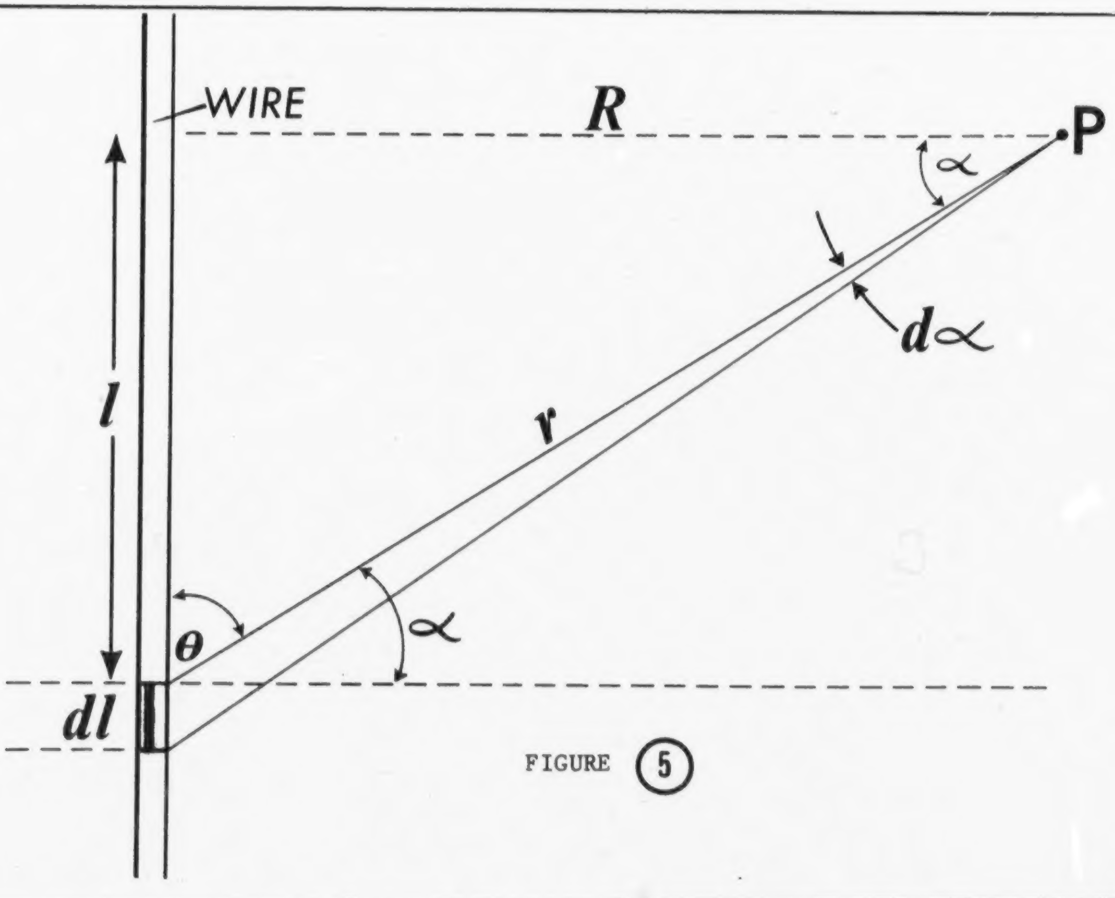

WIRE
R
P
α
dα
r
l
θ
α
dl

FIGURE ⑤

Figure 6 illustrates how the Biot-Savart Law may be set up in terms of α rather than θ. Since α is the complement of θ, $\sin \theta$ is merely replaced by $\cos \alpha$.

To perform this integration, it is of course necessary to express all of the quantities on the right side of the equation in terms of only one variable. In this particular approach to the problem, all these quantities are to be expressed in terms of α. Referring back to Figure 5 for a moment, note that the length of the wire $\underline{l}$ is related to the distance R by the tangent of the angle α. That is, one may write $l = R \tan \alpha$ since $\tan \alpha = l/R$.

Please refer now to Figure 7. Clearly, dl is needed in the equation; hence, 1 may be differentiated with respect to α to obtain it. This differentiation is shown in Figure 7 and should be studied carefully before proceeding.

The next task is to set up r in terms of α. Please refer to Figure 8. Since $\cos \alpha$ equals R divided by r, then $r = R/\cos \alpha$. To simplify the work it is better to express r in terms of the secant of the angle as given in Figure 8.

$$B = \int \frac{\mu_0}{4\pi} \frac{i\ dl\ \sin\Theta}{r^2}$$

$$B = \int \frac{\mu_0}{4\pi} \frac{i\ dl\ \cos\alpha}{r^2}$$

FIGURE ⑥

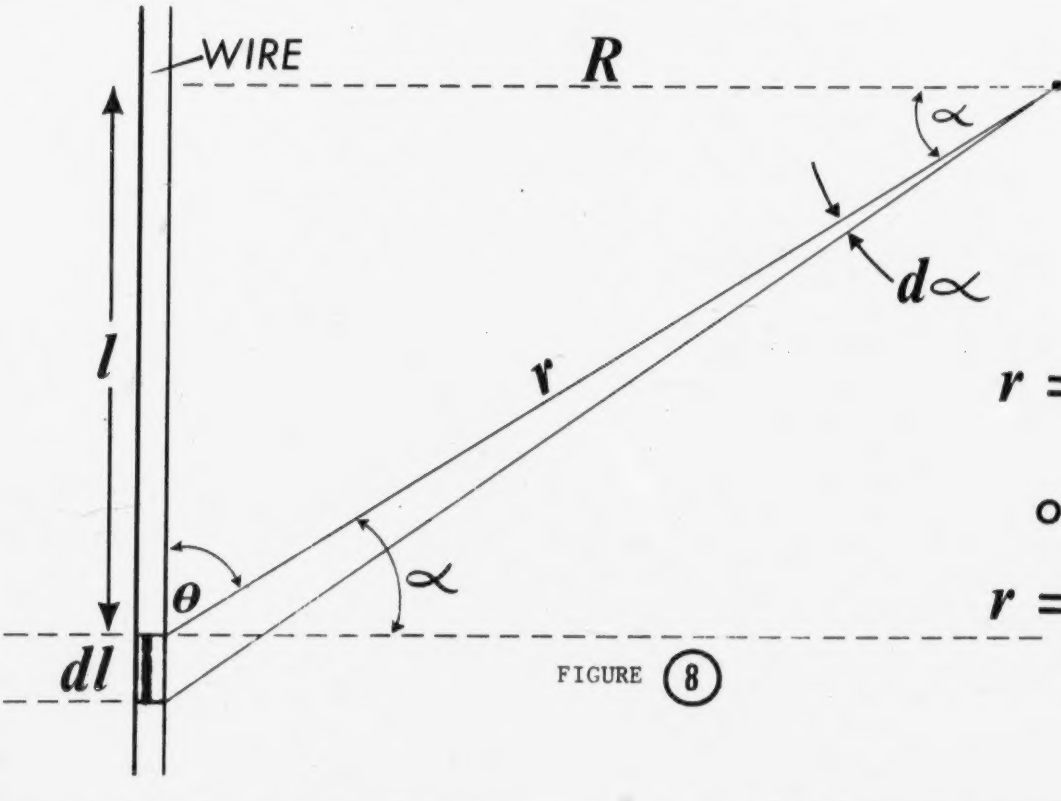
$$B = \int \frac{\mu_0}{4\pi} \frac{i\ dl\ \cos\alpha}{r^2}$$

$$l = R \tan\alpha$$

$$\frac{dl}{d\alpha} = R \sec^2\alpha$$

or $dl = R \sec^2\alpha\, d\alpha$

FIGURE ⑦



$$r = \frac{R}{\cos\alpha}$$

or better

$$r = R \sec\alpha$$

FIGURE ⑧

Figure 9 shows the step-by-step procedure used in substituting the trigonometric relationships into the general form of the Biot-Savart equation.

Step (a): $R \sec^2 \alpha \; d\alpha$ has been substituted for dl.
Step (b): $R^2 \sec^2 \alpha$ has been substituted for r.
Step (c): Simplification.
Step (d): Set up to integrate between chosen limits to find B at point P.

The integral of the cosine of an angle is the sine of the same angle. This is one of the reasons for selecting this approach: evaluation of the integral is extremely simple. Now, going to Figure 10, the solution is apparent. Please study this carefully. Note that the final expression for B is identical with the solution obtained by directly applying Ampere's Law to the same configuration.

The student will find that many problems can be easily solved by using the Biot-Savart Law while these same problems would be considerably more difficult if he attempts to apply Ampere's Law to them.

As a student, you have a significant advantage over a practising scientist. When a scientist encounters a practical problem, he cannot at the outset be sure that a solution for it exists, nor can he be certain that his mathematical tools and techniques are adequate for the job. On the other hand, the students may be quite certain that his Study Guide will not present insoluble problems, and that patience and care, plus the basic skills acquired by practice and study, will be enough to assure success.

(a) $$dB = \frac{\mu_0 i}{4\pi} \frac{\boxed{R \sec^2 \alpha} \cos \alpha \boxed{d\alpha}}{r^2}$$

(b) $$dB = \frac{\mu_0 i}{4\pi} \frac{R \sec^2 \alpha \cos \alpha \, d\alpha}{\boxed{R^2 \sec^2 \alpha}}$$

(c) $$dB = \frac{\mu_0 i}{4\pi} \frac{\cos \alpha \, d\alpha}{R}$$

(d) $$B = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dB = \frac{\mu_0 i_0}{4\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \alpha \, d\alpha$$

FIGURE ⑨

(a) $$B = \frac{\mu_0 i}{4\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \alpha \, d\alpha$$

becomes

(b) $$B = \frac{\mu_0 i}{4\pi R} \Big[\sin \alpha\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\mu_0 i}{4\pi R} \Big[1 - (-1)\Big]$$

(c) $$B = \frac{\mu_0 i}{2\pi R}$$

FIGURE ⑩

THE LAW OF BIOT-SAVART

TERMINAL OBJECTIVES

15/1 A Derive the expression for the magnetic induction within an ideal solenoid as (equation) is the actural current in the solenoid wire and n is the number of turns. (diagram)

15/1 D Use Fig. 4 as an aid in mathematically deriving the equation for the magnetic induction at point P; (equation).

FARADAY'S LAW OF INDUCTION

Faraday's Law, discovered by Michael Faraday in the 19th Century, represents still another great generalization of physics. Its sociological significance, too, ranks among the highest because many aspects of our modern technological civilization would have been greatly altered had this principle remained in obscurity.

A discussion of Faraday's Law properly requires that a few items of background material be briefly reviewed.

The magnetic flux ϕ across a surface is defined as the surface integral of the normal component of the magnetic induction B over the surface. Figure 1 presents the mathematical definition of magnetic flux which is clearly the parallel of electric flux with suitably altered symbolism. As the Figure indicates, the total flux across a given area is $\phi = \int \vec{B} \cdot d\vec{A}$. Since this is a dot product, the magnitude of the flux is related to the cosine of the angle between the normal to the plane of the surface and the actual direction of the B-lines. Referring to Figure 2, it is seen that the flux through the area can be found by integrating $B \cos \theta \, dA$ over the area under consideration. This idea should be studied for a while before proceeding.

$$d\phi = B_n\, dA \quad \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

$$\phi = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$$

FIGURE ①

$$\phi = \int B \cos \theta \; dA$$

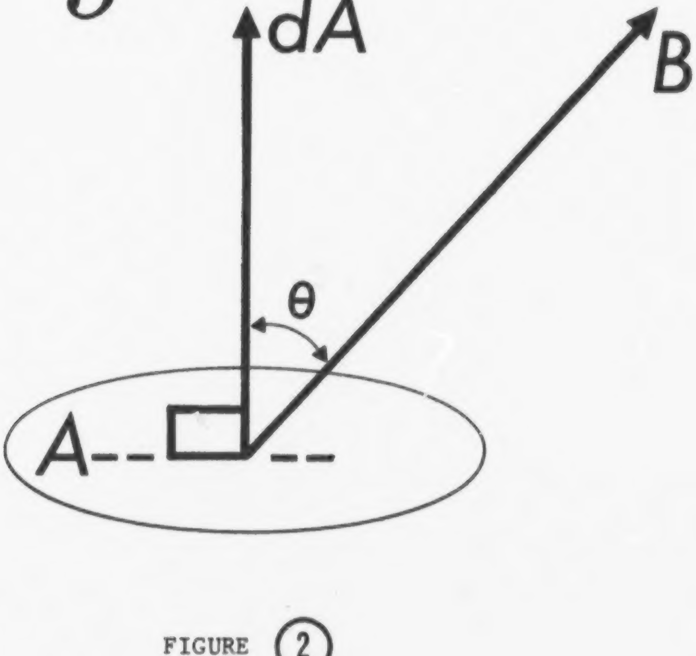


FIGURE ②

The physical outcomes of Faraday's Law are best approached through the medium of a simple experiment using the equipment and connections shown in Figure 3. A coil of insulated wire is connected to a sensitive galvanometer as shown. A magnet is held above the coil preparatory to inserting it into the coil. Although a galvanometer is a current-detector since a current must pass through its movement if a deflection of its needle is to be obtained, it may also be used to show the presence of an emf across its terminals. Initially, when there is no emf, the galvanometer needle is at a center zero position. When an emf is applied, the direction of needle deflection serves to indicate the direction of the current and, thus, the direction of the applied emf.

The magnetic field around a bar magnet may be visualized as lines of magnetic induction as in Figure 4. In general, these lines can be pictured as forming complete loops running roughly through the north-seeking and south-seeking ends of the magnet. At any point in the field near one of the magnetic poles, the field strength may be judged by the density of the lines at that point. The flux through a given area very close to the end of the magnet, for instance, would be substantially greater than the flux further away from the same end, through an equivalent area. This is evident from the way in which the lines spread out at greater distances from the end of the magnet.

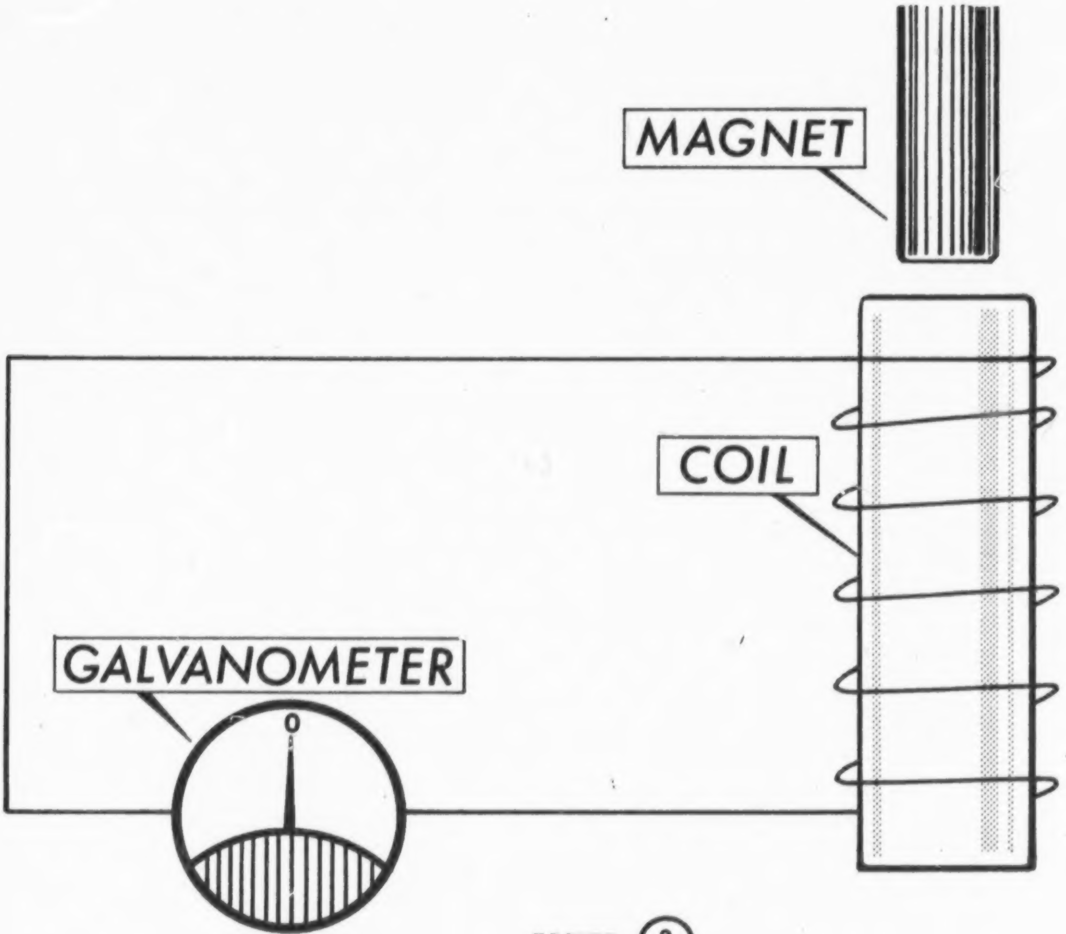

MAGNET
COIL
GALVANOMETER
0

FIGURE ③

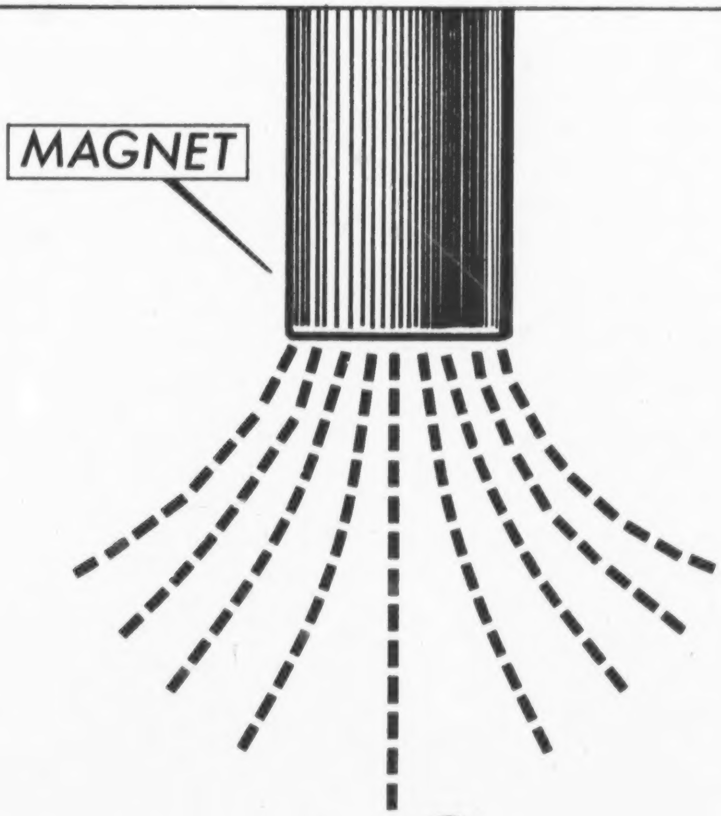


FIGURE ④

To perform the experiment, one end of the bar magnet is slowly inserted in the coil while the galvanometer is observed. A deflection occurs, indicating the presence of an emf across the meter terminals. This is displayed in Figure 5. In the illustrated example, a north-seeking or N-pole is moved downward into the coil causing the needle to deflect to the left. After insertion, the magnet is brought to rest inside the coil; at this time, the galvanometer reading drops back to zero. When the magnet is slowly removed from the coil, a deflection is again observed, this time in the opposite direction, toward the right as shown in Figure 6.

The same experiment may be performed in a slightly different manner by moving the coil relative to a stationary magnet starting with both at rest. When this is done -- say when the coil is moved upward with respect to the stationary N-pole inside it -- the galvanomenter again deflects. The direction of the deflection in this case is the same as it was when the N-pole was moved upward in the previous case. In short, it is the direction of the relative motion of the coil and magnet which appears to be the important factor in determining the sense of the current.

In the second phase of the experiment, a comparison is made between the amount of deflection obtained as a function of the speed of the motion, that is, a comparison of the induced emf for fast relative motion and slow relative motion. It is observed that the magnitude of the deflection increases with increasing speed of relative motion.

The question naturally arises at this point: what interaction is taking place? What causes the emf to be induced? Apparently relative motion of coil and magnet results in a change in the amount of flux that cuts through the conductors of the coil. Regardless of the way one performs this experiment, it is always found that the magnitude of the emf, and hence the magnitude of the current in the galvanometer, depends upon the time rate of change of the flux.

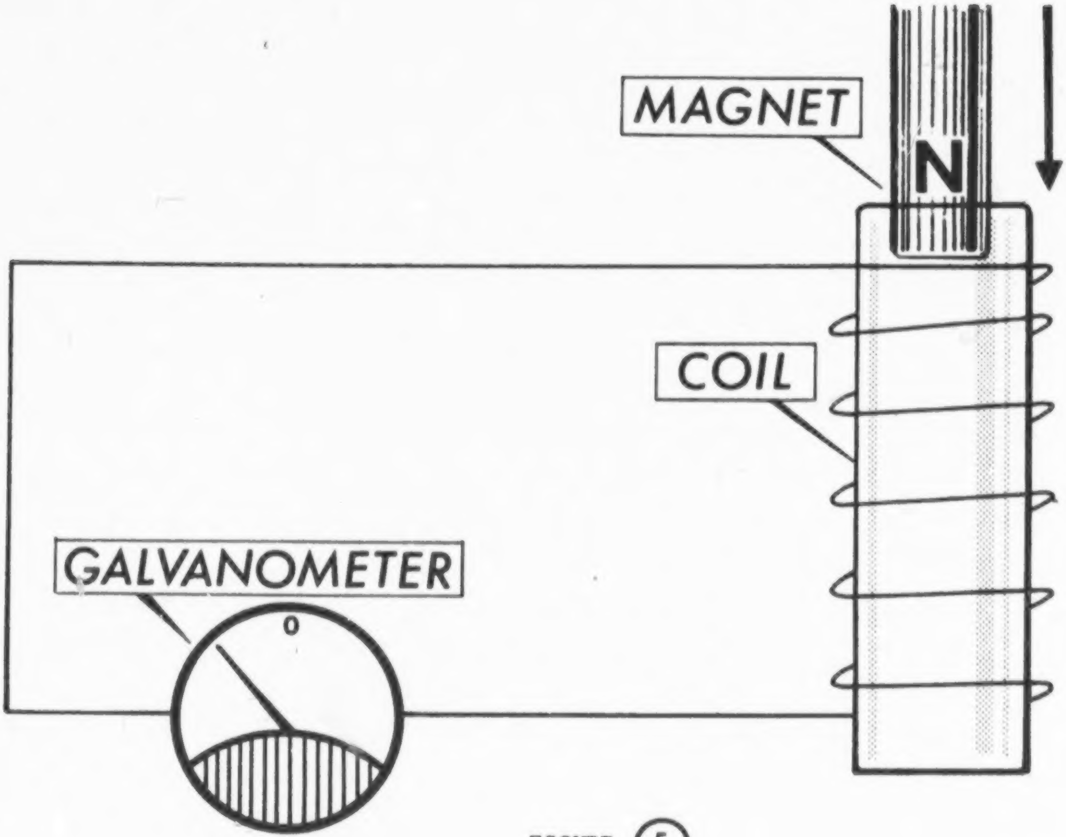

MAGNET
N
COIL
GALVANOMETER
0

FIGURE ⑤

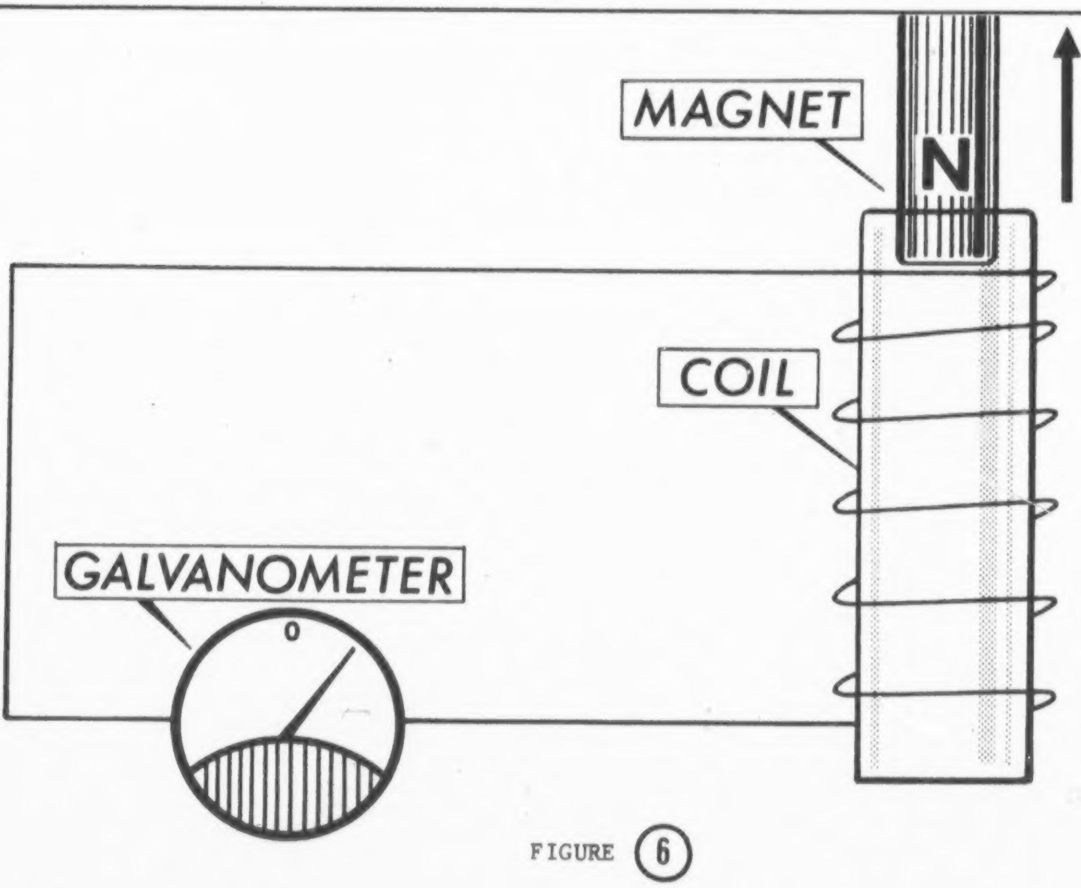

MAGNET
N
COIL
GALVANOMETER
0

FIGURE ⑥

This key opposition may be dramatically demonstrated by means of another simple experiment as illustrated in Figure 13. A coil of many turns is wound on a long iron core and connected to a suitable seat of emf through a spring pushbutton. A seamless aluminum ring rests on the coil with the core passing through it. Aluminum is used for two reasons: first, it is not a magnetic material and displays no ferromagnetic properties. Hence, any magnetic phenomena we might observe in connection with the ring cannot be blamed on its material. Second, aluminum has a very low electrical resistance so that even a small emf induced in the ring can cause a large current around its circumference. When the switch is closed (pushbutton depressed), an increasing flux builds up in and around the core causing the flux cutting through the aluminum ring to undergo a very rapid time rate of change. Although the flux build-up is not instantaneous, it does occur so swiftly that $d\phi/dt$ assumes an enormous value. The induced emf is correspondingly great and because the resistance of aluminum is so small, the induced current is immense. Thus, the newly induced magnetic field around the ring is very, very large.

This tremendous induced field must oppose the causative field; the repulsive force thereby developed must therefore be relatively great. The result is that the ring is thrust upward and away from the coil so that it flies away from the system straight up into the air. When done properly, this experiment is quite spectacular. The ring can be made to fly upward with enough initial velocity to strike the ceiling of the room with a resounding thwack.

$$\mathcal{E} \propto \frac{d\Phi}{dt}$$

FIGURE ⑦

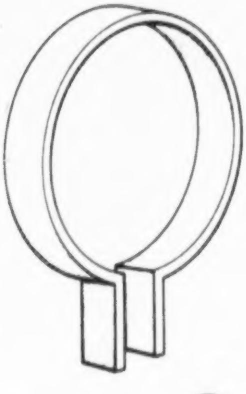


FIGURE ⑧

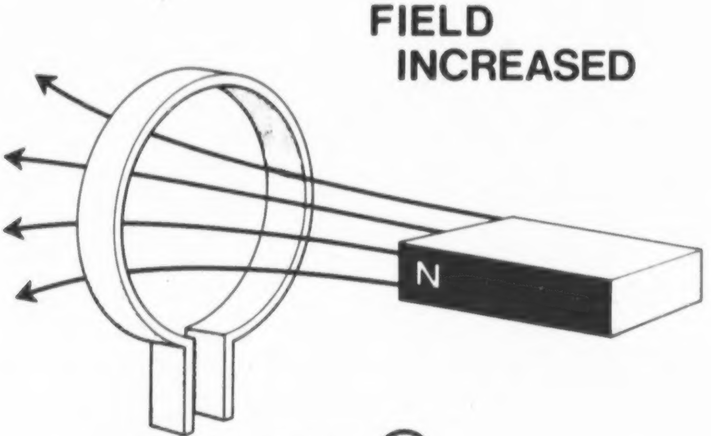
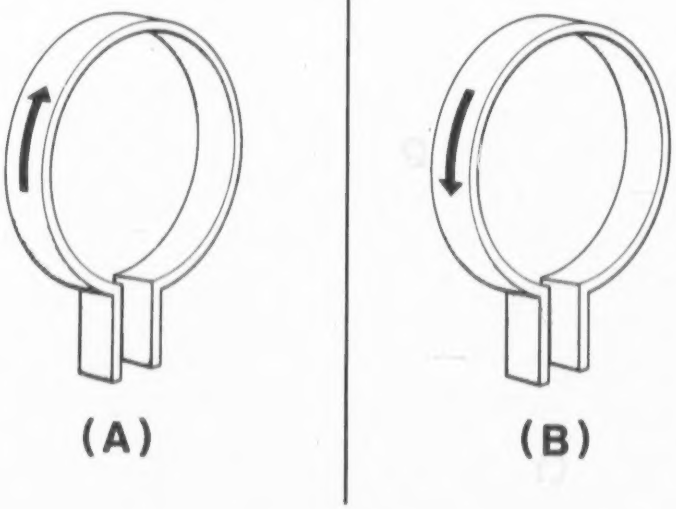

FIELD
INCREASED
N

FIGURE ⑨



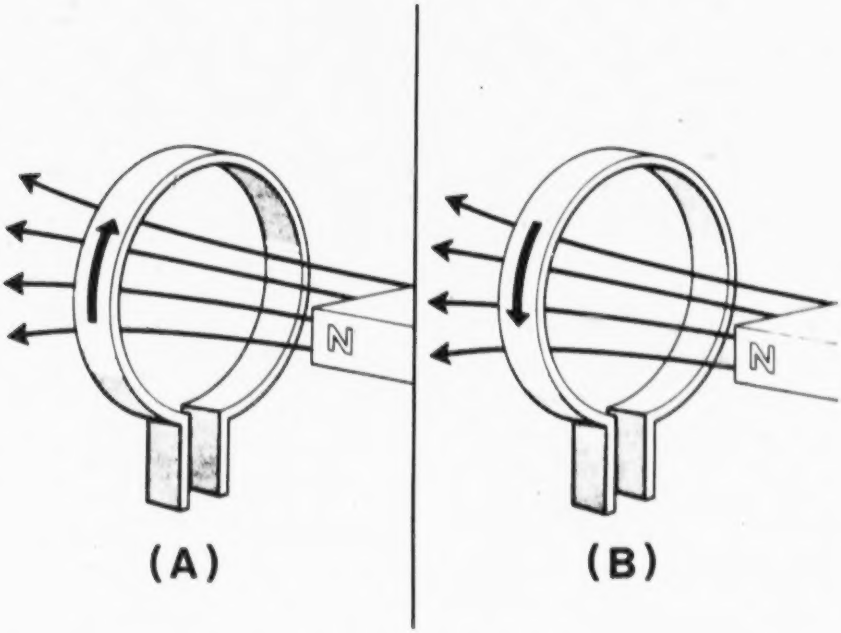
(A) (B)

FIGURE ⑩

Thus, an emf induced in a conductor as a result of relative motion of the conductor in a magnetic field has a magnitude that depends on $d\phi/dt$ as shown in Figure 7. This relationship is given as a proportion, that is, the induced emf symbolized by the script "E" is proportional to the time rate of change of the magnetic flux, $d\phi/dt$.

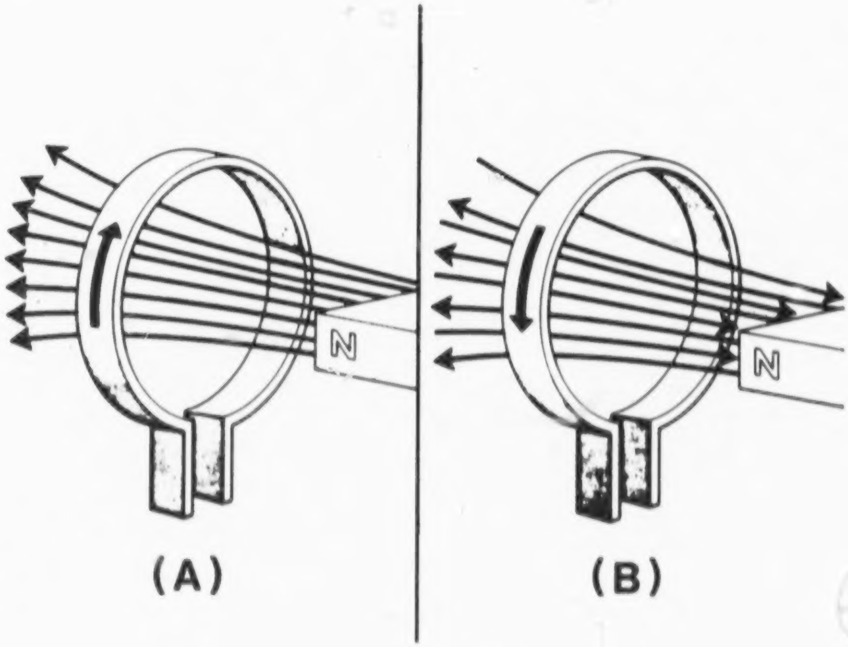
It is now possible to develop a logical scheme for determining the direction of the induced emf when other necessary things are known. Here again, a sensible approach is to study a specific case and then apply what is learned about this specific case to a valid generalization.

Figure 8 shows a metallic loop of fixed area; a galvanometer or some other indicator of induced emf is imagined to be connected to the ends of the loop. The loop is next moved toward an N-pole of a bar magnet as in Figure 9. As the relative motion proceeds, the flux through the loop increases since it moves through a region of greater flux density as it approaches more closely to the magnetic pole. As was previously shown, a change of flux results in an induced emf which in turn causes a current in the closed circuit of the loop and galvanometer. It is necessary now to determine whether or not the direction of this induced emf can be predicted from an analysis of all the other relevant factors. There are only two possibilities, both of which are illustrated in Figure 10: the direction of the induced current will be either clockwise as in Figure 10A or counterclockwise as in B. The fact that there is a current in the loop, regardless of its direction, means that a new magnetic field has come into existence -- the field produced by this current. Its direction is easily established by using the right-hand rule: grasp the loop with the fingers of the right hand encircling the loop in the direction of the current; the extended thumb will then point in the direction of the magnetic field.



(A) (B)

FIGURE ⑪



(A) (B)

FIGURE ⑫

Consider case A first. The right hand rule shows that the field due to the induced current will thread through the loop from right to left as in Figure 11. In case B, the same rule gives the direction of the field from left to right.

Figure 12 has superimposed the two previous diagrams on one another so that the combined effects become visible. In each diagram, one field originates at the external bar magnet's pole while the other arises from the current induced in the loop. In case A, both the applied and induced fields have the same direction -- from right to left. Note the fundamental impossibility this implies. Increasing flux leads to increasing induced current which leads to increasing induced field which leads to increasing flux which leads to increasing induced current ----- and so on. This endless chain suggests the possibility of infinite induced currents and infinite fields, absurdities, of course. It is a flat contradiction of the principle of conservation of energy. For this reason alone, Case A must be discarded as a natural impossibility.

On the other hand, case B is quite possible because the applied and induced fields are oppositely directed. Since they oppose one another, there is no implicit nor explicit violation of the conservation principle. Case B must, therefore, show the situation as it must exist in nature.

With opposing fields taken as being the true nature of things, it is then clear that the induced emf is not only proportional to $d\phi/dt$ but also that it is equal to the negative of $d\phi/dt$. This may be stated as follows: the direction of the induced emf must be such as to produce a current whose magnetic field opposes the change of flux which initially induced the emf.

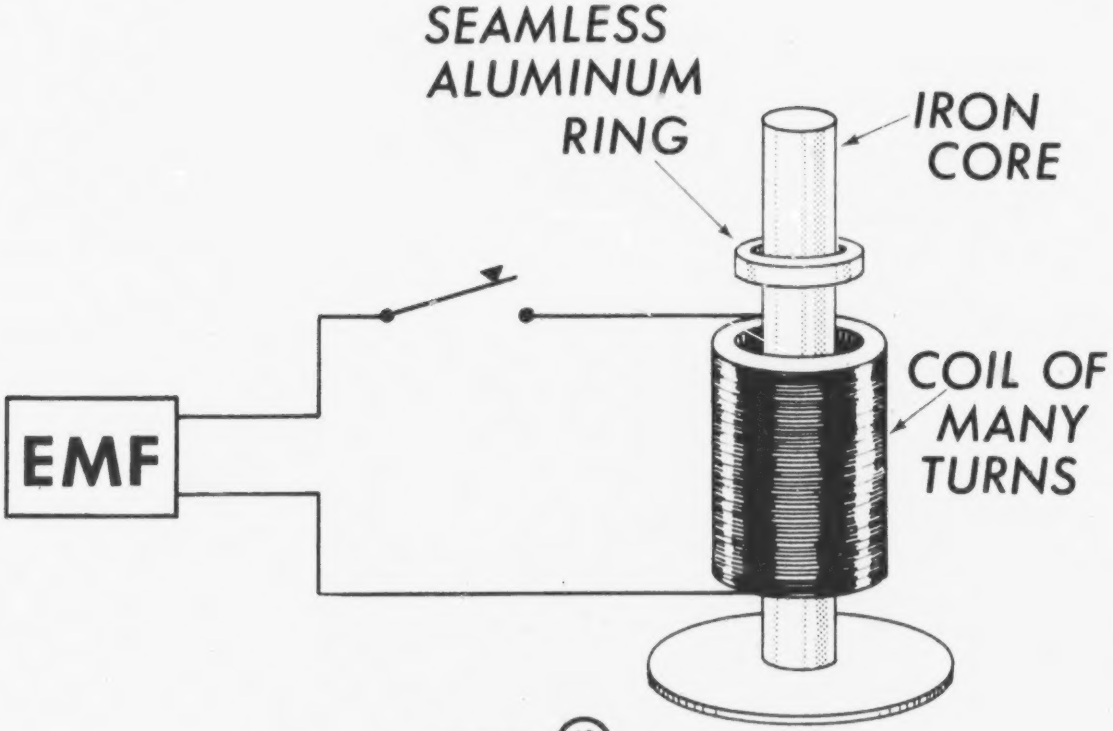

SEAMLESS ALUMINUM RING
IRON CORE
COIL OF MANY TURNS
EMF

FIGURE ⑬

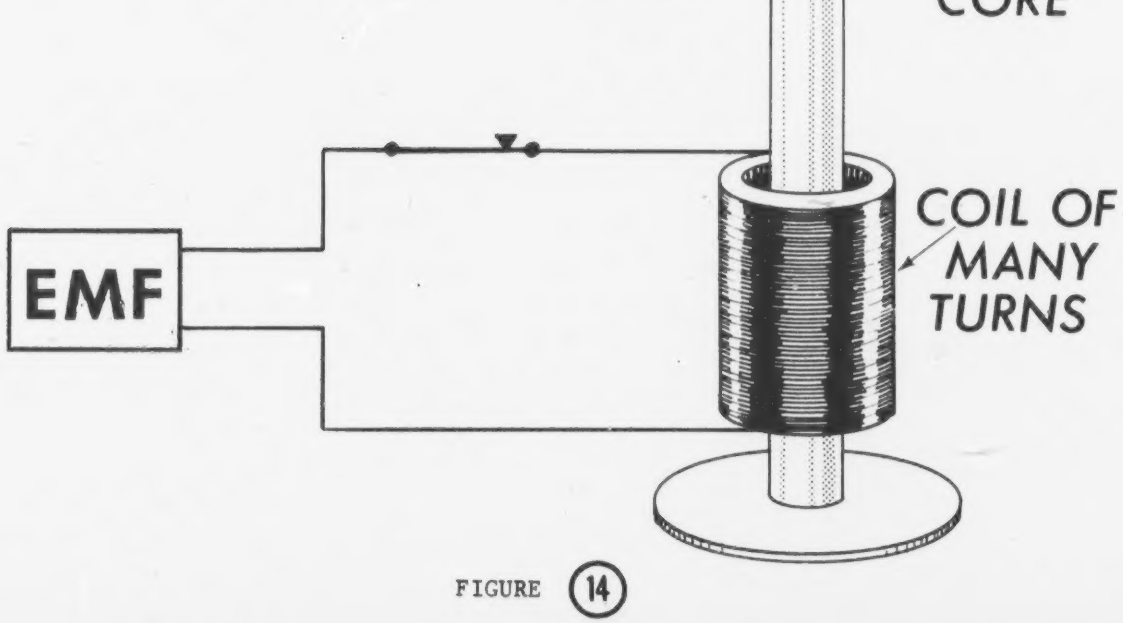

SEAMLESS ALUMINUM RING
IRON CORE
COIL OF MANY TURNS
EMF

FIGURE ⑭

The descriptive aspect of Faraday's Law involving the direction of the induced emf (and current) is generally known as Lenz's Law. This is indicated in the summary presented in Figure 15. In studying the summary, please note that the two laws are very intimately related -- one is quite valueless without the other.

FARADAY'S LAW

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

LENZ'S LAW

The direction of an induced current is such as to oppose the change of flux causing it.

FIGURE ⑮

FARADAY'S LAW OF INDUCTION

TERMINAL OBJECTIVES

15/3 A Trace the development of Faraday's Law of electromagnetic induction through an analysis of his basic experiments.

15/3 D Apply Lenz's Law to determine the direction of induced emf's in various induction situations.

MOTION OF AN ELECTRON IN COMBINED **E** AND **B** FIELDS

The beginning of the twentieth century witnessed a number of important experiments which marked the inception of modern physics. Not the least among these were the brilliant investigations into the nature and characteristics of the electron performed by the English scientist, Sir Joseph John Thomson. By studying the effect of combined electric and magnetic fields on moving electrons, Thomson determined for the first time the charge-to-mass ratio (e/m) of the electron. The apparatus described in this text resembles Thomson's equipment very closely; the cathode-ray tube used in a prior discussion ("Deflection of Electrons in an Electric Field") is to be applied again, this time to an analysis of the motion of an electron beam in a combined electric and magnetic field.

(Figure 1) The cathode-ray tube shown in this drawing has been previously presented but a brief review would not be out of place here. Electrons sprayed from the hot cathode are focused by the cylinder surrounding the heater-cathode assembly and accelerated by the anode adjacent to it. The electrons pass in the form of a beam through the small opening in the anode and proceed in a collimated pencil to the fluorescent screen at the end of the tube. The fluorescent spot marks the terminus of the beam at the screen. If there is no difference of potential between the parallel plates in the path of the beam, the electrons pass through without deviation. When a potential difference is present, however, the beam is deviated to an extent determined by the magnitude of the voltage and the geometry of the tube: the direction of the deviation, that is to the left or to the right, is governed by the direction of the electric field set up by the potential difference between the plates.

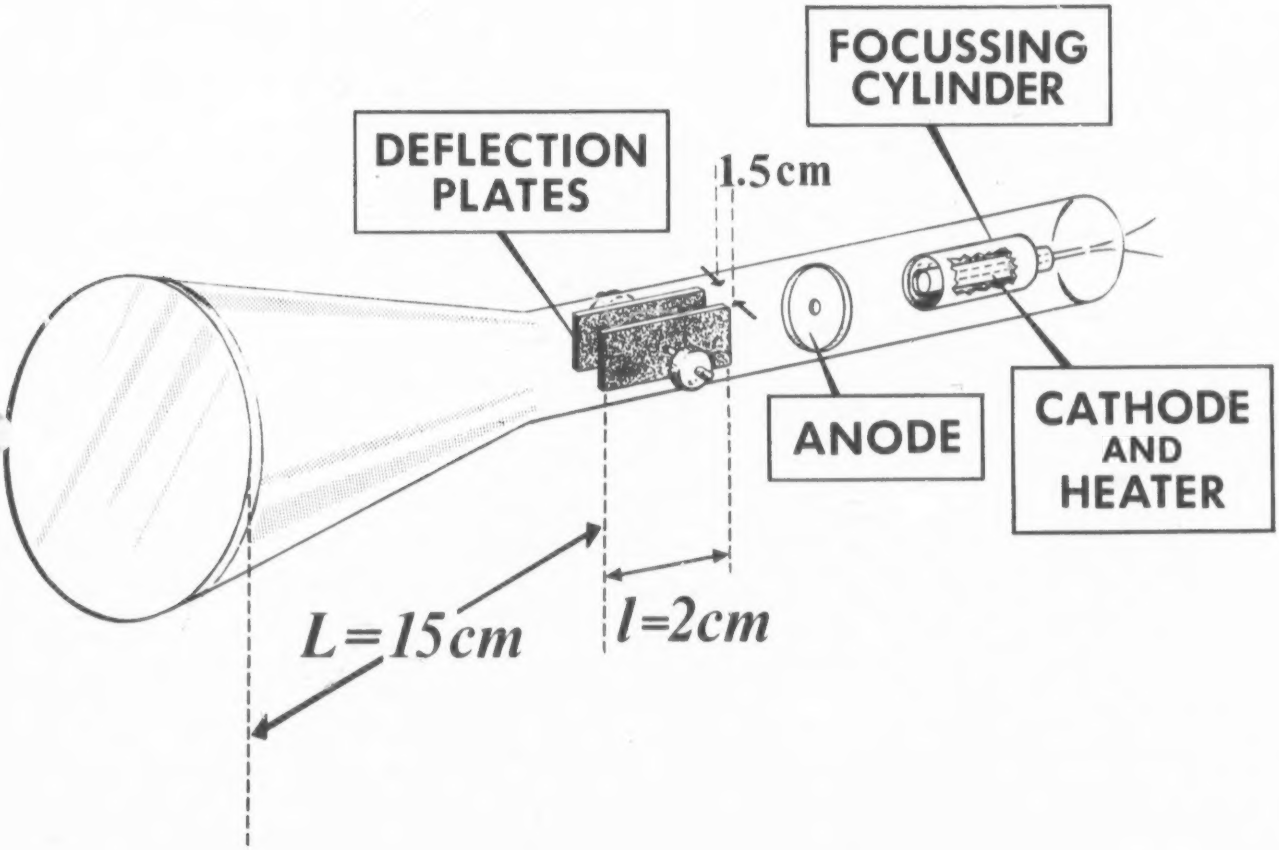

FOCUSSING CYLINDER
DEFLECTION PLATES
1.5cm
ANODE
CATHODE AND HEATER
L=15cm
l=2cm

FIGURE ①

(Figure 2) This is a top plan view of the edges of the deflecting plates. With the polarity of potential difference as shown -- the right-hand plate positive (viewed from the observer's position facing the screen) and the left-hand plate negative -- the electron beam is deflected toward the right. It must be remembered that the beam consists of negatively charged particles, hence the force acting will be in a direction opposite that of the field. The field is directed toward the left, the electrons are deflected toward the right.

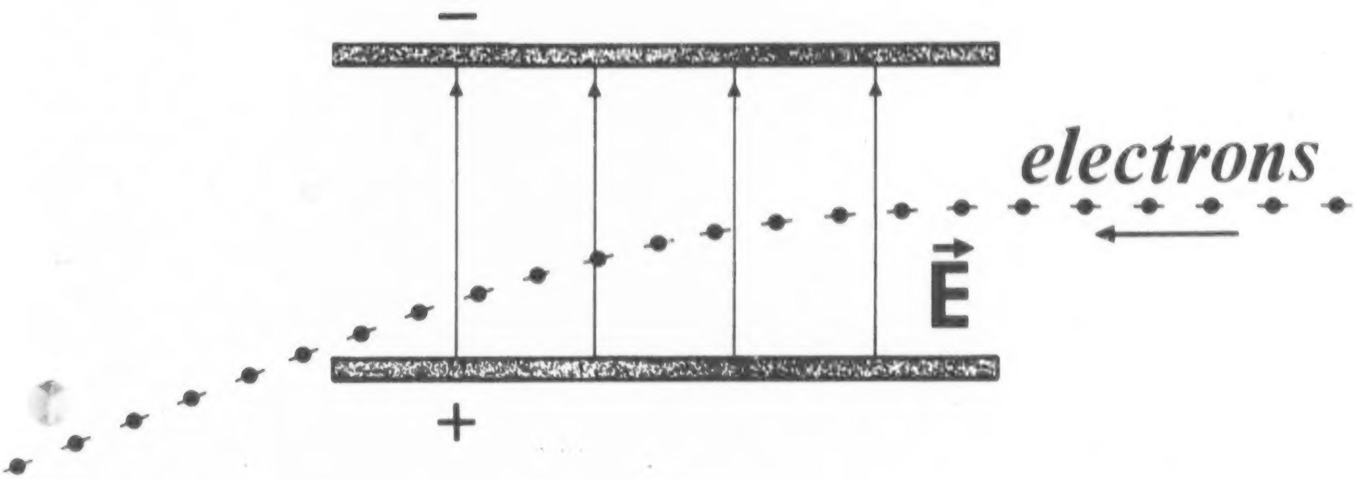

−
electrons
$\vec{E}$
+

FIGURE ②

(Figure 3) Electrons are also deflected when they pass through a magnetic field provided that some component of the electron velocity is perpendicular to the field. In this drawing a positive particle q is shown moving upward with velocity $\vec{v}$ at right angles to a field directed from right to left, $\vec{B}$. The right-hand Palm Rule indicates that the force experienced by q is directed toward the observer as shown, and is perpendicular to the plane containing $\vec{B}$ and $\vec{v}$. The equation in the figure also gives the vector equation for the force: it shows that the force is a cross-product in which $\vec{v}$ is rotated into $\vec{B}$. This is the relationship of particle velocity, B-field direction, and force for a positive particle. The change required in the relationship when the particle is negative is shown in Figure 4.

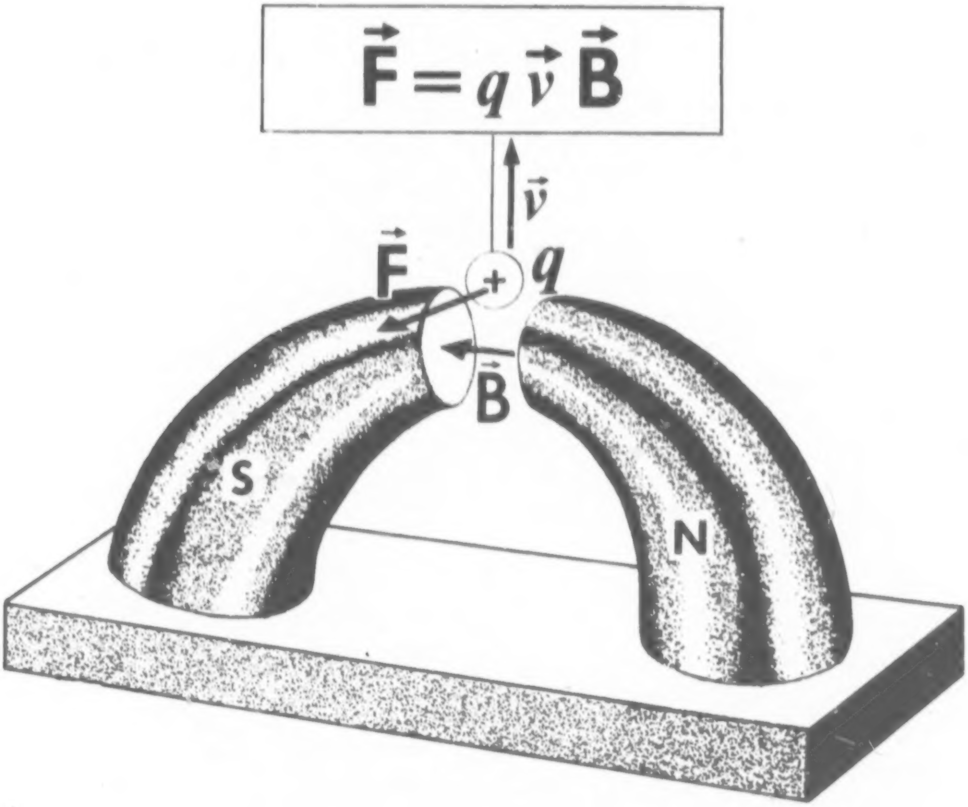

$\vec{F} = q\vec{v}\vec{B}$
$\vec{v}$
$\vec{F}$
+
q
$\vec{B}$
S
N

FIGURE ③

(Figure 4) For a negative particle like an electron, the force $\vec{F}$ is still a cross-product in which $\vec{v}$ is rotated into $\vec{B}$, but this time it is multiplied by negative e, the charge on the electron. Thus, the force is 180° from the direction it had when the particle was positive. The same result is obtained when the left-hand Palm Rule is used. Note, however, that the force remains perpendicular to the plane containing $\vec{v}$ and $\vec{B}$.

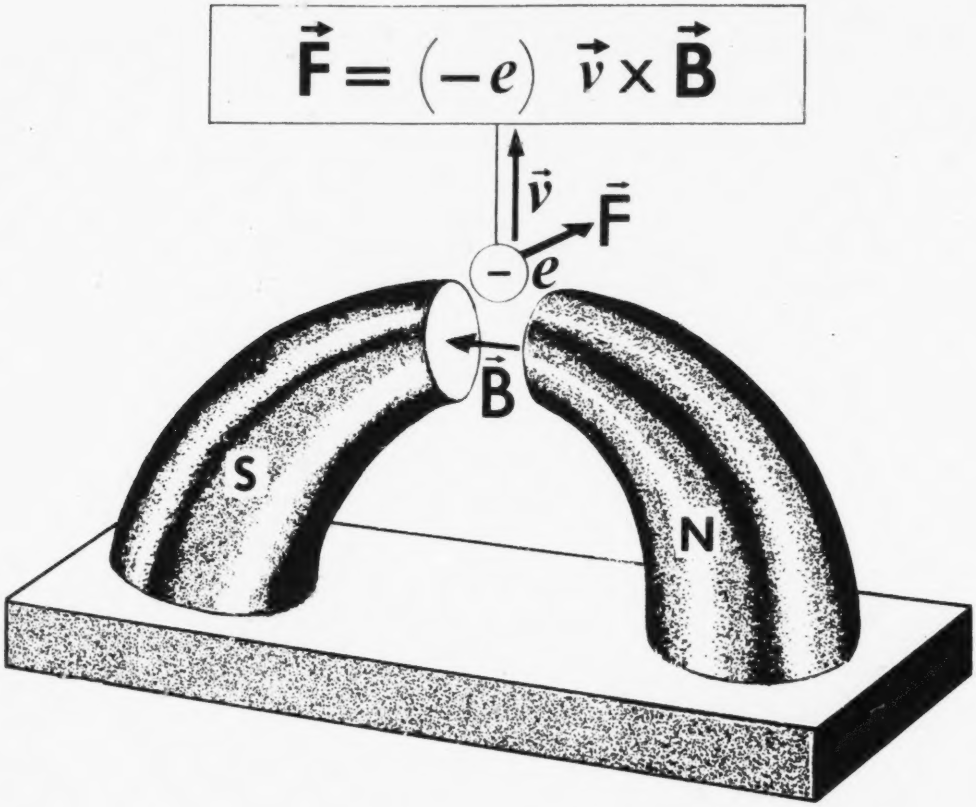

$\vec{F} = (-e)\ \vec{v} \times \vec{B}$
$\vec{v}$
$\vec{F}$
$-$
e
$\vec{B}$
S
N

FIGURE ④

(Figure 5) In the equipment under discussion, the B-field is developed by a current-carrying coil placed above the tube in line with the deflection plates. In this case, the current direction is chosen so that the B-field is directed downward. The student should now apply the left-hand Palm Rule to verify the direction of the force on the electron beam as it is given in the drawing: fingers of the left hand pointing downward, extended thumb in the direction of the velocity $\vec{v}$, force toward the left as viewed by an observer standing in front of the screen. Note that things have been arranged so that the B-field gives rise to a deflection in a direction opposite that of the B-field discussed before: left for the B-field, right for the E-field. The extent to which the beam is deflected is readily controlled by the operator by suitably changing the electrical values. The B-field magnitude may be altered by changing the current in the coil and the E-field may also be varied by changing the potential difference across the deflection plates.

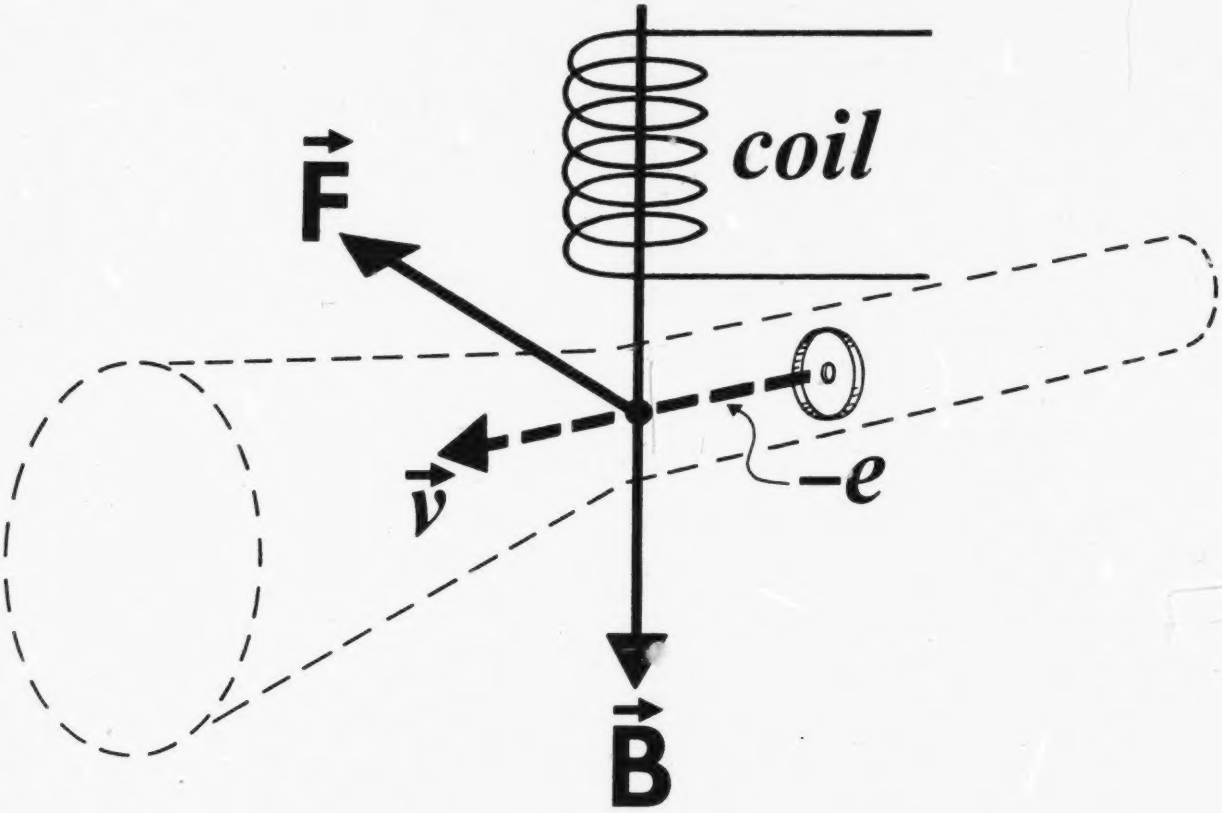

$\vec{F}$
coil
$\vec{v}$
$-e$
$\vec{B}$

FIGURE ⑤

(Figure 6) This is a diagram of the front screen and the fluorescent spots in positions obtained for special electrical conditions. On the left is the spot position for a specific B-field in which the force acting on the beam is $F_B = evB$. On the right is the spot position for specific electric field which exerts a force $F_E = eE$. In the center is the spot position when F_B is equal to F_E, both forces acting for the same length of time on the beam. Stated otherwise, the beam is undeflected when $F_B = F_E$ over the same time interval of action. Thus, a properly selected B-field can nullify the deflection caused by a given E-field, or vice versa. In actual practice, the E-field potential is selected to produce a deflection of a few centimeters and then the current in the coil is adjusted until the spot returns to its undeviated position. For this balanced condition, the forces may be equated. The student is asked to set the equivalents of F_B and F_E equal to one another and then solve for $\vec{v}$, the velocity of the electron beam. This should be done before turning to the next page.

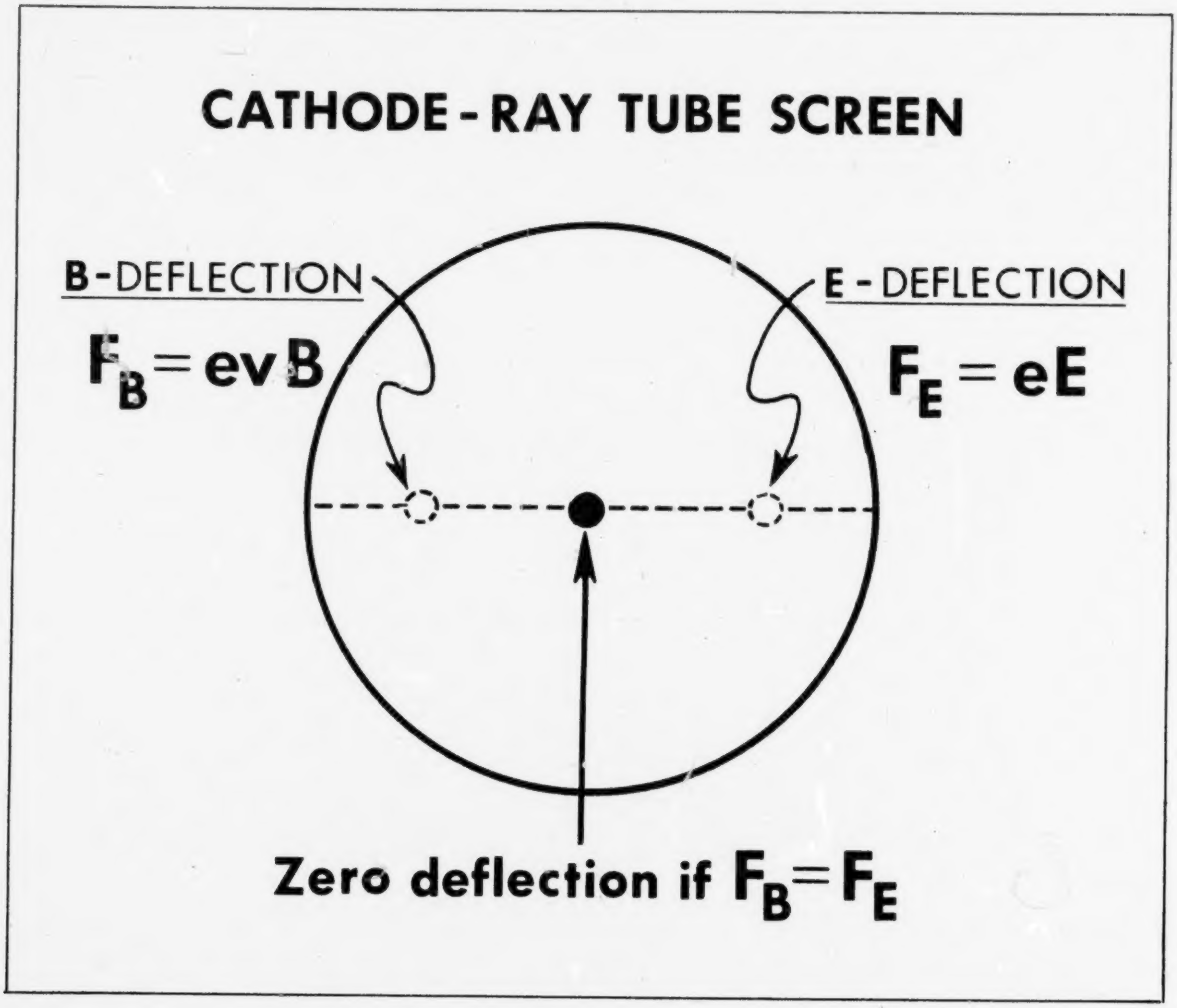

CATHODE - RAY TUBE SCREEN
B - DEFLECTION
$F_B = evB$
E - DEFLECTION
$F_E = eE$
Zero deflection if $F_B = F_E$

FIGURE ⑥

(Figure 7) The solution is shown here. It is seen that the beam velocity may be calculated very simply from the ratio of E to B. This provides an easy method for measuring the velocity of the beam since both E and B are readily measurable individually. With the velocity known, calculation of e/m then becomes a matter of applying straightforward, elementary mechanics to the geometry of the tube. Various approaches may be used, all of them depending on the assumption that the beam velocity can be measured. Many of these procedures are fully described in elementary college textbooks.

$$evB = eE$$

$$vB = E$$

$$v = \frac{E}{B}$$

FIGURE ⑦

MOTION OF AN ELECTRON IN COMBINED E AND B FIELDS

TERMINAL OBJECTIVES

10/3 B Answer questions and solve problems relating to potential field strength.

14/1 B Answer qualitative questions relating to the magnetic induction vector $\vec{B}$.

L - R
TRANSIENTS

L-R TRANSIENTS

A transient electric current is a current of temporary nature which appears in conductors as a result of the transfer of stored energy somewhere in the circuit. In the study of transients in circuits containing resistance and capacitance, it was shown that such energy transfers cannot occur instantaneously. A specific time is required for a transient current to grow or decay. It was further demonstrated that the delay time in either case is a function of the magnitudes of the circuit constants and that growth and decay times can be calculated by applying the relevant mathematical relationships.

Circuits containing inductance $\underline{L}$, resistance $\underline{R}$, and a seat of emf $\underline{\mathcal{E}}$ also display delay phenomena. Just as an RC circuit has a time constant, a circuit containing $\underline{L}$ and $\underline{R}$ may be shown to have a similar characteristic which governs the time required for a current in it to grow to some desired value, or to decay from some initial value to some other lower one.

This paper deals with the development of the relationships relevant to the growth and decay of transient currents in L-R circuits.

Figure 1: This is a schematic diagram of a common laboratory set up designed to show that L-R transients do indeed exist. The coil must be a large one containing many turns of relatively heavy wire. The switch S_1, the light bulb on the left, the switch S_2, and the entire coil are all connected in series. The light bulb at the left is connected across part of the coil.

When both switches are closed simultaneously, both lamps light but the growth of the current is substantially slower than it would be if the lamps had been connected directly to the 100-volt DC generator. The delay effect is readily observable, particularly if the coil is properly wound. As in RC circuits, the delay phenomenon is explained in terms of energy transfer: the current from the generator gives rise to the growth of a magnetic field in and around the coil. The electrical energy is converted into energy stored in the magnetic field and, since energy cannot be transferred instantaneously, a finite time is required for the current to grow from its initial to its final magnitude. Essentially, the inductance of the coil impedes the growth of the current in the circuit.

Assuming that both switches have been closed for an interval long enough to allow the current to reach some maximum value, the effect of opening switch S_2 is then observed. Two things are then seen to occur: as the switch opens, a violent electrical arc appears across it, vanishing only after the switch has been opened all the way; secondly, the right-hand lamp flashes on so brightly that it may very well be destroyed. From the circuit point of view, opening switch S_2 disconnects the voltage source from the circuit, allowing whatever energy that has been stored in the inductor to dissipate itself in the form of a current in the right-hand lamp. Just before S_2 is opened, a strong magnetic field is present in the coil; when the voltage source which maintains this field is disconnected, the field collapses and induces an emf in the coil itself. The current resulting from this induces emf has only one path to take --- through the associated light bulb. Since the stored energy cannot be transferred instantaneously, the effect is that the coil tends to keep the current flowing for a considerable time after the switch has been opened. The current maintained by the collapsing field when the switch is opened is an L-R transient which may give rise to potential differences that are very much greater than the source voltage. It is often more dangerous to turn off the current in a coil than it is to turn it on!

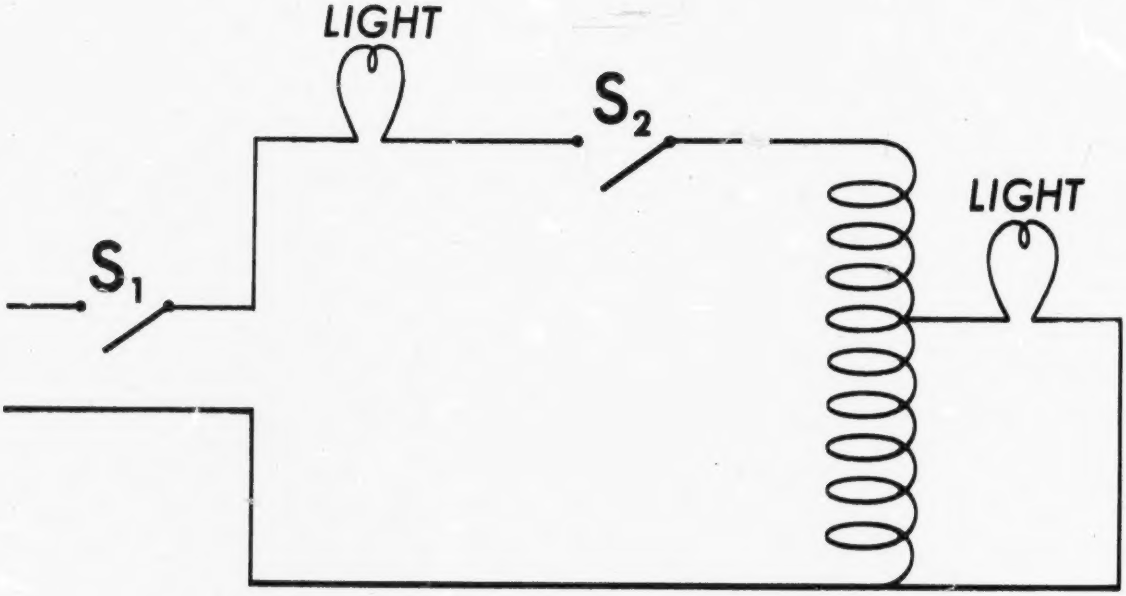

LIGHT
S2
S1
LIGHT

FIGURE ①

Figure 2: The method used to analyze L-R transients is similar to the one employed in the study of RC circuits. To begin the analysis, consider that the switch S in the circuit shown here is open and that a current i has been established in the resistance and the coil. The switch is then closed, short-circuiting the source of emf. (This is never done in practice because it would damage both the source and the switch. The circuit is drawn this way to avoid unnecessary complexities). As the magnetic field in the coil collapses, it induces a transient current around the closed loop containing the switch, the resistance, and the coil. The duration of the transient current depends on the circuit constants as has already been mentioned, but its instantaneous value i may be used in setting up a Kirchhoff loop equation.

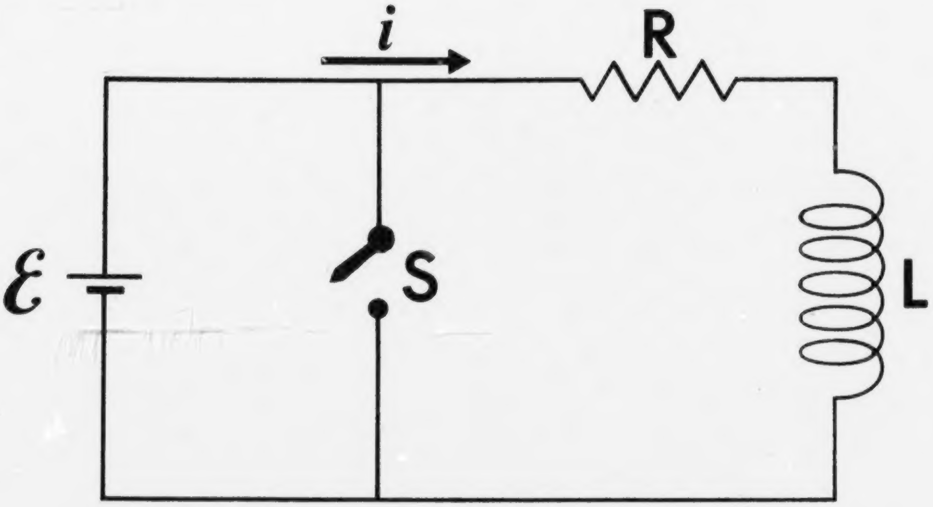

i
R
$\mathcal{E}$
S
L

FIGURE ②

Figure 3: The first expression is the required loop equation. The first term, Ri, is the voltage drop in the resistance due to the presence of the instantaneous current i. The second term in this equation is the voltage induced across the coil as the magnetic field collapses, that is, the inductance multiplied by the rate of change of the current di/dt. Since this traversal completes the loop, the sum of these voltage drops is set equal to zero as indicated.

The second equation shows a rearrangement of terms in which Ri has been shifted to the right side and has had its sign changed.

The third expression is a second rearrangement of terms to group the variables together in preparation for the required integration.

$$Ri + L\frac{di}{dt} = 0$$

$$L\frac{di}{dt} = -Ri$$

$$\frac{di}{i} = -\frac{R}{L}dt$$

FIGURE ③

Figure 4: The final expression obtained in the previous figure is repeated at the top of this group. Taking indefinite integrals yields the second equation of this group. It should be remembered that the constant of integration may be expressed in any form desired; in this case, the subsequent steps are facilitated by using the logarithm to the base e of a constant number, "ln constant".

Antilogs are then taken to establish the third equation of the group in which the instantaneous current i appears to be equal to some constant multiplied by e raised to the -Rt/L power. To find the value of the constant, the time t is set equal to zero so that the entire exponent becomes zero. Thus $e^0 = 1$ so that the constant must be equal to the instantaneous current at zero time, that is, at the instant of closing the switch. This permits us to write the final expression of the group showing that the instantaneous current i at any time t is a function of the initial current, the resistance R, and the inductance L.

At this point, in preparation for the next step, it would be helpful to review one small aspect of the RC time constant briefly.

$$\frac{di}{i} = -\frac{R}{L}\,dt$$

$$\ln i = -\frac{R}{L}\,t + \ln(\text{constant})$$

$$i = (\text{constant})\,e^{-Rt/L}$$

$$i = i_o\,e^{-Rt/L}$$

FIGURE ④

Figure 5: The first expression shown here is the exponent of e in the RC transient equation. Note that t is in the numerator and is isolated from the other circuit quantities which appear in the denominator. In this form, the product RC serves a very useful purpose as the time-constant, a quantity of significance in circuit design.

As it turns out, it is advantageous to modify the exponent of e in the L-R expression so that it, too, contains only the factor t in the numerator with the L and R factors in the denominator. This modification is shown in the second equation of the group.

When set up in this form, L/R has an analogous significance as the time-constant of L-R circuits. It may be readily shown that L/R has the dimension of time and may be expressed in seconds. This is left as an exercise for the student. (Hint: set up a ratio of the henry to the ohm, convert these units to their fundamental forms, then solve).

$$-{}^{t}/_{RC}$$

$$-{}^{Rt}/_{L} = -{}^{t}/_{\frac{L}{R}}$$

$$\frac{L}{R} = \textit{time constant}$$

FIGURE ⑤

Figure 6: To recapitulate, the expression obtained for the decay of current in an L-R circuit is:

$$i = i_o e^{-Rt/L}$$

$$\text{OR} \quad i = i_o e^{-t/\frac{L}{R}}$$

The analysis of current growth in an L-R circuit is handled in much the same manner. With the source of emf connected to the series arrangement of L and R, the current starts at zero and begins to increase as it builds up the magnetic field in the coil. The Kirchhoff loop equation for the instantaneous current $\underline{i}$ at any time $\underline{t}$ is given by the first expression in this group where $\underline{\mathcal{E}}$ is the source emf. Since this development is quite similar in concept to the one discussed in R-C TRANSIENTS, it will be left as an important and valuable exercise for the individual student. Fill in the intermediate steps between the first and second expressions shown here and note that L/R is once again the time-constant of the circuit.

The final equation shows that the instantaneous current $\underline{i}$ after an interval of current growth $\underline{t}$ is equal to i_∞ multiplied by $(1 - e^{-Rt/L})$. The term i_∞ is the current that would appear in the circuit after an infinitely long interval of current growth.

$$Ri + L\frac{di}{dt} = \mathcal{E}$$

$$i = i_\infty\left(1 - e^{-Rt/L}\right)$$

$$\frac{L}{R} = \text{time constant}$$

FIGURE ⑥

Figure 7: Here is a summary of the previous discussion. A great deal can be learned about the significance of these equations by substituting various values for t, R, and L and noting how the instantaneous current changes. For example, determine the fraction of the initial current in the decay equation that would be present in the circuit after an interval of one time-constant period, that is, where t = R/L. Test the equations for t = 0 and t = infinity.

Bear in mind that i_0 in the decay equation is the initial current before decay starts and that it is given by $\mathcal{E}/R$.

Note, too, that i_∞ is the current that would be present in the circuit after an infinite growth time. This value of the current is also given by $\mathcal{E}/R$.

CURRENT DECAY

$$i = i_o \, e^{-Rt/L}$$

CURRENT GROWTH

$$i = i_\infty \left(1 - e^{-t/\frac{L}{R}} \right)$$

For Both

$$i_o = \frac{\mathcal{E}}{R}$$

$$i_\infty = \frac{\mathcal{E}}{R}$$

FIGURE ⑦

L - R
TRANSIENTS

TERMINAL OBJECTIVES

15 03 124 00	analyze the general RL current growth equation qualitatively and quantitatively.
15 03 126 00	analyze the general RL current decay equation qualitatively and quantitatively.

R-C
TRANSIENTS

R-C TRANSIENTS

The word "transient" as it is used in physics implies something temporary just as it does in ordinary English. A transient in a hotel, for example, is a temporary resident, one who rents a room for a day or two. In physics, the term implies a lack of permanency or a condition which is the opposite of a "steady state". A transient current in an electrical circuit is a current that arises because of a potential difference between two points of a conductor but lasts for a relatively short time. A transient electric current is readily demonstrated with the simple equipment shown in Figure 1.

Figure 1: This is a schematic diagram of a circuit containing a seat of emf (a battery), an ordinary incandescent lamp, a single-pole double throw switch, and a capacitor. With the capacitor initially uncharged and the switch in the lower position, the capacitor circuit is open and there is no current. When the switch is moved to the upper position, the circuit is complete so that the battery begins to charge the capacitor and current appears in the conductors. As the capacitor charges and the potential difference across its terminals increases, the current in the circuit gradually decreases because the polarity of the capacitor voltage is in opposition to that of the battery. After a time, no further current can be detected -- it has died out. Thus, this is a transient current which persists only as long as the capacitor has not charged to its maximum voltage. The action is made visible by the incandescent lamp. When the switch is first moved up, the lamp flashes on brightly but then begins to dim as the transient current starts to die out. The time required for the decay of the transient current is equivalent to the charging time of the capacitor. Energy cannot be instantaneously transferred from source to receiver in any natural phenomenon; in this case, a capacitor cannot change its state of charge in either direction instantaneously. When the switch is moved down, the lamp again flashes on, remains on for approximately the same time as before, and then goes out. Once more, a definite time is needed for the capacitor to transfer its stored energy to the light bulb.

CHARGE - DISCHARGE CIRCUIT

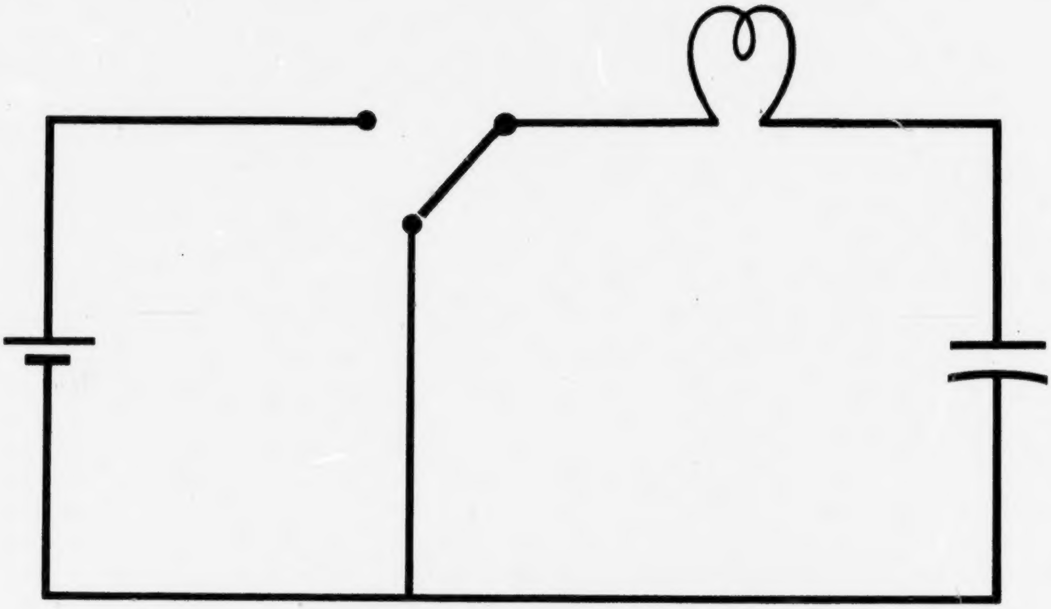


FIGURE ①

Figure 2: Looking into this analytically, it is assumed at the start that the capacitor has been fully charged and that the switch has just been moved to its lower position, starting the discharge process. At this instant, the Kirchhoff loop equation can be written as shown here. Starting at the top plate of the capacitor and going around the loop in a counterclockwise direction, the first component encountered is the lamp of resistance R carrying an instantaneous current i so that the voltage drop across it is Ri. The capacitor is next in line in the traversal of the loop; here is found a potential difference due to the charge on the capacitor. The voltage is, of course, given by the ratio of charge to capacitance or q/C. Back at the starting point of the loop, the voltage is equal to the initial value at the beginning of the traverse so that the sum of the two voltage drops must equal zero as indicated in the first equation.

In the second equation, dq/dt has been substituted for the instantaneous current i.

Transposing terms yields the third equation. This may be read verbally as: the rate of change of charge of the capacitor at any instant is numerically equal to the initial charge q on the capacitor divided by the product of the resistance in the circuit and the capacitance of the capacitor. This product RC is known as the time constant of the circuit. Its significance will be demonstrated shortly.

$$Ri + \frac{q}{C} = 0$$

$$R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$\frac{dq}{dt} = -\frac{q}{RC}$$

FIGURE ②

Figure 3: This differential equation is not difficult to solve. The terms of the first equation are rearranged as shown to obtain the second.

The indefinite integral of each side is then taken to obtain the third equation. That is, the integral of dq/dt is simply ln $\underline{q}$, and the indefinite integral of -dt/RC is -t/RC plus a constant. Since the constant may be chosen in any form desired, it is easier in this case to write it as the logarithm of a constant, or ln (const).

Figure 4: The next step involves taking the antilog of the expression developed previously, here shown as the upper equation. When this is done, the second equation is obtained. To determine the value of the constant, $\underline{t}$ is set equal to zero so that the entire exponential term becomes zero. This means that, for this assumption, $\underline{e}$ becomes unity. Hence, when $t = 0$, the constant is equal to the initial charge q_o and the expression takes the form shown in the lowermost equation.

$$\frac{dq}{dt} = -\frac{q}{RC}$$

$$\frac{dq}{q} = -\frac{dt}{RC}$$

$$\ln q = -\frac{t}{RC} + \ln\left(constant\right)$$

FIGURE ③

$$\ln q = -\frac{t}{RC} + \ln\left(constant\right)$$

$$q = \left(constant\right)\ e^{-t/RC}$$

$$q_0 = constant$$

$$q = q_0\, e^{-t/RC}$$

FIGURE ④

Figure 5: From the dimensional point of view, if the right side of the first expression shown here is to have the same units as the left side, the term $e^{-t/RC}$ must be dimensionless. It follows, therefore, that the product RC must have the same dimension as t, that is it must be expressed in time units. It is not difficult to show that this is indeed true: RC is measured in seconds and provides an indication of the rate at which the transient current decay of the discharging capacitor occurs. Essentially, this is the reason for referring to RC as the time constant of the circuit. If this product is made larger by increasing either the resistance, the capacitance, or both, the decay-time increases correspondingly in accordance with the relationship shown here. It should also be noted that, in order for a capacitor to discharge fully -- actually to zero -- the time constant must theoretically, at least, be infinite. In practice, however, a capacitor is considered to be fully discharged after an elapsed time of five time-constant periods. For example, in a circuit containing a capacitance of 1.0 microfarad and a resistance of 1.0 megohm, the time constant is 1 second. When such a capacitor is allowed to discharge from some initial value for a period of 5.0 seconds, the voltage across its terminals is then taken to be zero; it is then considered to have discharged fully.

$$q = q_o e^{-t/RC}$$

$RC =$ *time constant*

FIGURE ⑤

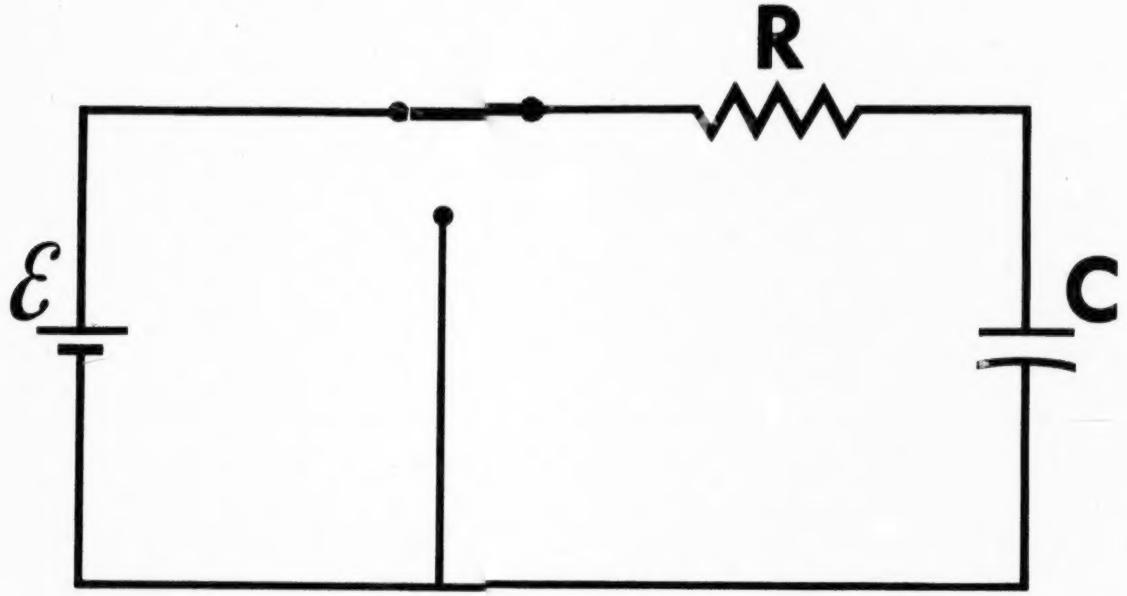
Figure 6: The relationship for the charging transient is obtained in a similar manner although the development is somewhat more complicated. To avoid unnecessary mathematical complexities, an indication of the method of obtaining the equation for charge will be presented rather than a rigorous derivation.

The switch is moved to the upper position to start the charging process, assuming that the capacitor has been previously fully discharged. The first equation shows the Kirchhoff loop relationship for this situation. The addition of the seat of emf mandates the inclusion of the "E" term on the right side.

Figure 7: This equation can be solved by finding the complementary function and adding to it the interval in which we are interested. It has already been shown that the solution takes the form:

$$q = (\text{const})\ e^{-t/RC}$$

The constant is then added to obtain the third equation shown here. If this equation is substituted back into the first, it is readily shown (rather tediously, however) that the last equation of this group is the result. The student is urged to analyze this development thoroughly for himself.



$$R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}$$

FIGURE ⑥

$$R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}$$

$$R\frac{dq}{dt} + \frac{q}{C} = 0$$

$$q = (constant)\ e^{-t/RC} + B$$

$$q = q_{\infty}\ \left(1 - e^{-t/RC}\right)$$

FIGURE ⑦

Figure 8: Here is a summary of the two equations for transients that have been developed:

DISCHARGE: A capacitor has been charged to its maximum value, q_o. This is its initial charge. The equation then states that the instantaneous charge q remaining on the capacitor at any time t after discharge has begun is given by the right-hand term.

CHARGE: A capacitor has been fully discharged. A seat of emf is then connected to it through a resistance. Then, the instantaneous charge q after a charging time t is related to the charge the capacitor would have assumed if allowed to charge for an infinite time is given by the right-hand term.

Thus, in the discharging phenomenon, q_o is the initial full charge or maximum charge that can be taken on given enough time. In the charging phenomenon, q_∞ is the charge the capacitor would have assumed had it been given infinite time to do so. In either case, q_o or q_∞ can be replaced by EC as shown in the last statement.

DISCHARGE:

$$q = q_0\, e^{-t/RC}$$

CHARGING:

$$q = q_\infty \left(1 - e^{-t/RC}\right)$$

$$q_0 = q_\infty = \mathcal{E} C$$

FIGURE ⑧

Figure 9: A widely-used application of the R-C transient effect is shown schematically in this diagram. A source of direct current such as a battery is connected across a capacitor and a gas-filled tube such as a neon or argon lamp through a series resistor. The observed effect when this circuit is in operation is a periodic flashing of the lamp.

The explanation is best started by considering the instant at which the capacitor is fully discharged, the voltage across the neon lamp is zero, and the lamp is unlit. As the battery begins to charge the capacitor due to the transient current, the voltage across the lamp and capacitor starts to rise at a rate determined by the time constant RC. The exponential increase of voltage is illustrated in the graph.

A gas-filled glow tube is characterized by the fact that no light is visible when the potential difference between its terminals is below the required "breakdown" or ionization voltage. For a standard night-light type of lamp, this is approximately 60 volts. Thus, no effect is observed during the charging process until the voltage across the parallel combination grows to 60 volts. When this does occur, the gas ionizes and glows brightly. Simultaneously, the internal resistance of the lamp drops to a very low value. Since the lamp is connected directly across the capacitor, the latter is discharged very quickly by the conductive gas causing the voltage across the capacitor to drop correspondingly. At about 55 volts, the gas in the lamp deionizes and the lamp extinguishes. Its resistance again rises to its initial high value. Thus, the capacitor once more starts to charge until it again reaches the ionization potential of the lamp and the process repeats. The repetition rate of the flashing light is clearly governed by the time required for the voltage across the capacitor to build up from the deionization potential to the ionization potential. For a given lamp, the difference between these two potentials is nearly constant, hence the frequency of the flashes is governed by the RC time constant of the circuit. Altering R or C or both will therefore result in a changed frequency; increasing the RC product increases the period and decreases the repetition rate of the flashes, and vice versa.

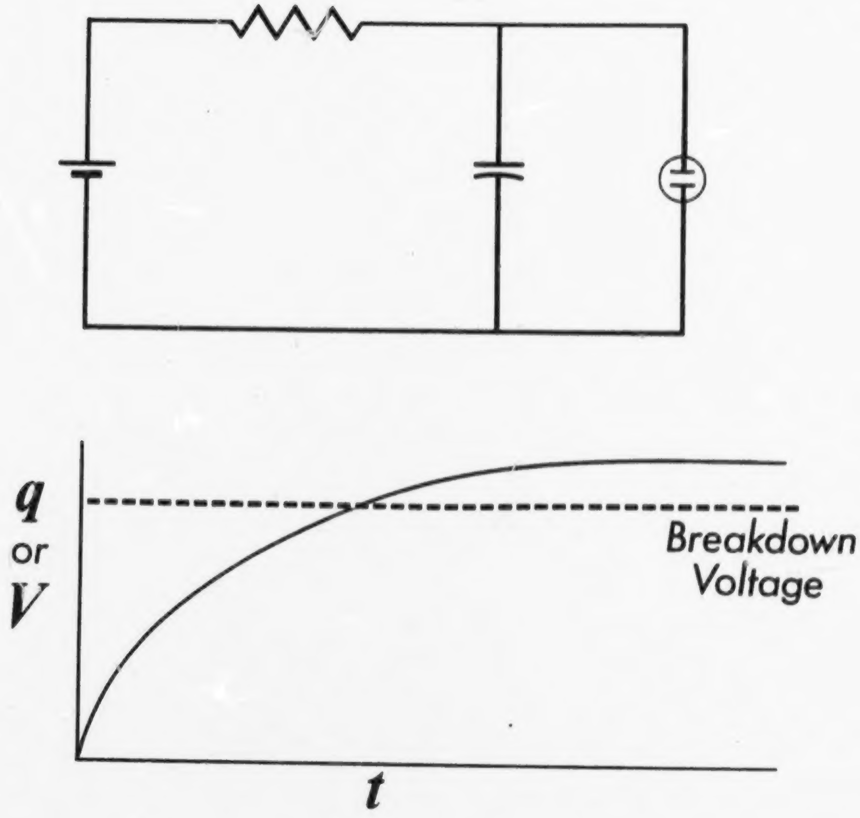

q
or
V
Breakdown
Voltage
t

FIGURE ⑨

R-C TRANSIENTS

TERMINAL OBJECTIVES

15 02 121 00 analyze the general RC circuit charging equation qualitatively and quantitatively.

15 02 123 00 analyze the general RC circuit discharge equation qualitatively and quantitatively.